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Optically induced band structure of free electrons in an external plane wave field

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Abstract. The motion of a non-relativistic free charged particle in a classical electromagnetic plane wave field is investigated. By introducing the ansatz of a modulated plane wave for the wavefunction, the problem is reduced to the solution of the general Mathieu equation for a circularly polarised field and to the solution of the Hill equation for a linearly polarised field. The corresponding eigenvalue equations show that physical values of the energy are separated into bands. The relationship with earlier methods is also discussed.

1. Introduction

In connection with the development of high power lasers, in recent years much effort has been devoted to the experimental and theoretical investigation of the interaction of an intense radiation field with matter. The present paper deals with the theoretical aspects of the interaction of non-relativistic free electrons with an external plane electromagnetic field. This problem originally started with the exact solution of the Dirac equation for a plane electromagnetic wave under specially chosen initial conditions (Volkov 1935). This paper did not, however, call much attention to itself because the appropriate light source was not available at the time. With the invention of the laser, theoretical interest has been renewed, and different exact and approximative solutions have been given by several authors, choosing other special initial conditions and methods of description (relativistic versus non-relativistic, quantum mechanical versus quantum electrodynamical). A review of the extended work performed in the 60's in this field is given by Eberly (1969). Here we just briefly recall the main steps leading to analytical expressions of the wavefunction. Shortly after the Volkov solution appeared, another exact solution of the Dirac equation with somewhat different initial conditions was presented and applied to the Compton scattering in intense fields (Alperin 1944). The solution of the Klein–Gordon equation of a relativistic scalar charged particle in an external field was given by Brown and Kibble (1964). The non-relativistic problem was concerned with using the corresponding Schrödinger equation (Nickle 1966). In Nickle's paper an 'almost' exact solution of the Schrödinger equation for a non-relativistic free electron interacting with an external field was given. Exact solutions were obtained only in the dipole approximation (Keldysh 1965, Kohler 1966). The same problem was reinvestigated from a different point of view, and analytical results were obtained beyond the dipole approximation only very recently (Ehlotzky 1978). In the last-cited paper a natural extension of the dipole result is given in analytical form. Reviews of recent analytical results are also given by the present authors (Bergou 1980, Bergou and Varró 1980). Nevertheless, in spite of the fact that exact analytical solutions exist for the classical equation of motion of a free charged particle in an external field, both relativistic and non-relativistic (Sengupta 1949, Vachaspati 1963), as well as for the relativistic quantum mechanical counterpart, by using either the Dirac or the Klein–Gordon equation, at present no analytical solution is available for the corresponding non-relativistic quantum mechanical problem. In this paper we proceed further along the lines followed by Ehlotzky and, in the following, by introducing some obvious physical assumptions, we show that the Schrödinger equation for the case under consideration can be reduced to the ordinary Mathieu differential equation. By considering the analytical properties of the solutions, it is found that the eigenvalues of the Mathieu equation are continuous at low intensities and form narrow bands at high intensities. Hence we conclude that the allowed energy eigenvalues also form bands as the intensity increases—by analogy with the formation of energy bands in the case of a periodic perturbation. We also propose experiments to observe this new effect.

2. Theory

In the following we consider the problem of the interaction of a non-relativistic charged free particle with an external plane wave field. The experiment we have in mind is the interaction of slow (e.g. thermally emitted) electrons with an intense light pulse (e.g. a mode-locked picosecond wavetrain). The corresponding Schrödinger equation describing steady-state behaviour can be written in the following form:

$$\frac{1}{2m}\left(\hat{\boldsymbol{p}}-\frac{e}{c}\boldsymbol{A}(\nu)\right)^2\Psi=\mathrm{i}\hbar\frac{\partial\Psi}{\partial t} \tag{1}$$

where

$$\boldsymbol{A}(\nu)=A_0(\boldsymbol{e}_1\cos\nu+\boldsymbol{e}_2\sin\nu),\qquad \nu=\omega(t-\boldsymbol{n}\boldsymbol{x}/c). \tag{2}$$

Here e is the charge, m the mass of the free particle; $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are polarisation vectors of the electromagnetic field, ω its frequency; $\boldsymbol{n}$ is the unit vector in the direction of the wave propagation. For the sake of simplicity we use a circularly polarised external field, because then $\boldsymbol{A}^2(\nu)=A_0^2$ is simply a constant. This assumption is not essential for, by using linearly polarised light instead of Mathieu's equation, one would arrive at Hill's equation, and the considerations about the band structure involved in the Mathieu solution remain essentially unchanged, but become slightly more difficult. After this explanation of the model, we introduce the following ansatz, i.e. we look for the solution of (1) in the form

$$\Psi(\boldsymbol{x},t)=\mathrm{e}^{(\mathrm{i}/\hbar)(\boldsymbol{p}\boldsymbol{x}-Et)}f(\nu),\qquad E=\boldsymbol{p}^2/2m, \tag{3}$$

the physical meaning of which is quite obvious. The wavefunction (3) describes a free particle modulated by the external field, and the modulation thus depends on the field variable ν only. After substituting (3) in (1), one obtains the following ordinary differential equation for $f(\nu)$:

$$\frac{\hbar\omega}{2mc^2(1-v_{\|}/c)}f''+\mathrm{i}f'+\frac{(e/mc)\boldsymbol{A}\boldsymbol{p}-e^2A_0^2/2mc^2}{\hbar\omega(1-v_{\|}/c)}f=0,\qquad v_{\|}=\boldsymbol{n}\boldsymbol{p}/m. \tag{4}$$

The coefficient of the second derivative is just the dimensionless ratio of the photon energy to the pair creation energy; therefore, in the case of optical frequencies its value is very small (of the order of 10^{-6}). Thus, if we neglect the first term on the LHS of (4), the remaining first-order equation can be solved exactly. Essentially this procedure was followed by Ehlotzky (1978), and by using the solution of this first-order equation in the calculation of the multiphoton inverse bremsstrahlung cross section, he obtained corrections to the well known formula obtained earlier in the dipole approximation. However, in the case of extremely high intensities the inclusion of the second derivative term can lead to new effects, for the following reasons. In (4) the ratio of the first derivative to the zeroth derivative is approximately equal to the ratio of the interaction energy to the photon energy:

$$\left|\frac{f'}{f}\right| \sim \frac{(e/mc)\boldsymbol{A}\boldsymbol{p} - e^2A_0^2/2mc^2}{\hbar\omega(1-v_{\parallel}/c)} \tag{5}$$

and the ratio of the second derivative to the first one is again the same. Therefore the magnitude of this ratio can compensate for the smallness of the coefficient of the second derivative. From here we may conclude that at very high intensities of the external field one has to retain all three terms on the LHS of (4) and look for an exact solution of it.

We may proceed in the following way. Notice that the coefficients of both f' and f'' are constant; therefore, by introducing the substitution

$$f(\nu) = \exp\left(-\mathrm{i}\frac{2mc^2(1-v_{\parallel}/c)}{\hbar\omega}\frac{1}{2}\nu\right)g(\nu), \tag{6}$$

the first derivative from (4) can be eliminated, and we arrive at the standard form of the Mathieu ordinary differential equation

$$g''(z) + (a - 2q\cos 2z)g(z) = 0 \tag{7}$$

where the following notations have been introduced:

$$a = \left(\frac{2mc^2}{\hbar\omega}\right)^2\left[\left(1-\frac{v_{\parallel}}{c}\right)^2 - \left(\frac{(e/c)A_0}{mc}\right)^2\right], \tag{7a}$$

$$q = -\left(\frac{2mc^2}{\hbar\omega}\right)^2\frac{(e/c)A_0}{mc}\frac{p_{\perp}}{mc}, \tag{7b}$$

$$p_{\perp} = [(\boldsymbol{e}_1\boldsymbol{p})^2 + (\boldsymbol{e}_2\boldsymbol{p})^2]^{1/2}, \tag{7c}$$

$$\alpha = \tan^{-1}[(\boldsymbol{e}_2\boldsymbol{p})/(\boldsymbol{e}_1\boldsymbol{p})], \tag{7d}$$

$$z = \tfrac{1}{2}(\nu - \alpha). \tag{7e}$$

Furthermore, it is easy to see that $p_{\perp}$ is a constant of motion, since the operators $\hat{p}_1 = \boldsymbol{e}_1\hat{\boldsymbol{p}}$ and $\hat{p}_2 = \boldsymbol{e}_2\hat{\boldsymbol{p}}$ commute with the original Hamiltonian. Therefore the parameter q is completely determined by the experimental conditions, and thus the only free parameter in (7) is $p_{\parallel} = \boldsymbol{n}\boldsymbol{p}$. The general solution of (7) can be written in the form (Arscott 1964)

$$g(z) = A\,\mathrm{e}^{-\mu z}\phi(z) + B\,\mathrm{e}^{\mu z}\phi(-z). \tag{8}$$

Here A and B are two constants, μ is the characteristic exponent and $\phi(z)$ is a π periodic function of z. The characteristic exponent can be given quite generally in the form $\mu = \delta + \mathrm{i}\beta$. By direct substitution of this expression into (8), one can observe that for $z \to \pm\infty$ the solution $g(z)$ becomes unstable, provided that $\delta \neq 0$. For a physically meaningful solution, therefore, we must require that δ vanish. If we substitute (8) into (7) we obtain the following eigenvalue equation for μ:

$$\cosh \pi\mu = 1 - 2\Delta(0) \sin^2 \tfrac{1}{2}\pi\sqrt{a} \tag{9}$$

where $\Delta(0)$ is a tridiagonal determinant of infinite order:

$$\Delta(\mathrm{i}\mu) = \begin{vmatrix} \cdot & \cdot & \cdot & & & \\ \gamma_{-2} & 1 & \gamma_{-2} & & & \\ & \gamma_0 & 1 & \gamma_0 & & \\ & & \gamma_2 & 1 & \gamma_2 & \\ & & & \cdot & \cdot & \cdot \end{vmatrix}$$

and

$$\gamma_{2r} = q/[(2r - \mathrm{i}\mu)^2 - a].$$

The equation (9) can be solved only numerically, and the solution gives the $\mu = \mu(a, q)$ function. For certain values of a,q $\delta = \mathrm{Re}\,\mu$ vanishes, and accordingly the a–q plane can be divided into stable and unstable regions. This stability diagram is represented in figure 1.

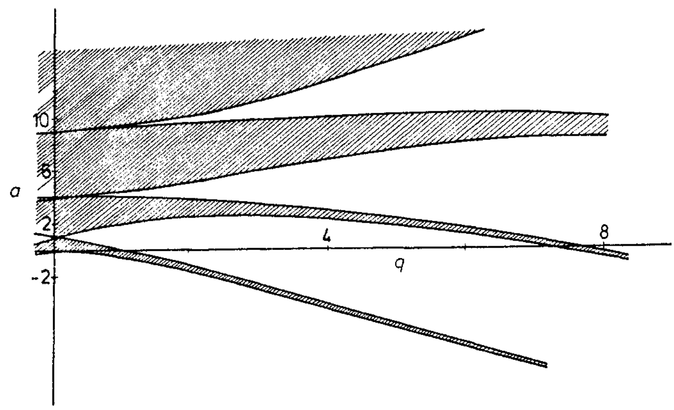

Figure 1. Stability diagram for Mathieu's equation. Stable regions are shaded; the rest of the a–q plane is unstable.

As q is entirely determined by the parameters of a given experiment—once the intensity and the frequency of the external field as well as the transverse momentum of the incoming electron, which is a constant of motion, are fixed, no free parameters in q are included—from figure 1 one can conclude that the possible a-values form bands. The only free parameter in the expression of a being $p_{\parallel}$, the possible $p_{\parallel}$ values also form bands. Also from figure 1, one can see that the possible a-bands become narrower and narrower as q increases, and in the limit of large q they can be approximated by the following analytical expression:

$$a = -2|q| + 2(2n+1)|q|^{1/2}. \tag{10}$$

Under usual experimental conditions the above expression holds almost completely, and by substituting here the expressions ($7a$) and ($7b$) for a and q we obtain the allowed

$p_\parallel$-values. The separation between neighbouring bands $\Delta p_\parallel = p_\parallel^{(n+1)} - p_\parallel^{(n)}$ can also be calculated easily, and for the relative separation we obtain finally the following approximate formula,

$$\left|\frac{\Delta p_\parallel}{p_\parallel}\right| = \frac{\hbar\omega}{mc^2} \frac{[(e/c)A_0 p_\perp]^{1/2}}{p_\parallel(1-p_\parallel/mc)}, \tag{11}$$

which is valid for high intensities and for $n \ll 2mc^2/\hbar\omega$, that is, the slow n dependence of the relative separation can be neglected. It is easy to show from (3) that the relative separation of the allowed energy bands is connected with (11) in a simple way:

$$\Delta E/E = 2\Delta p_\parallel/p_\parallel. \tag{12}$$

If, for example, the intensity of the light field is 10^{14} W cm^{-2} the relative energy separation becomes 10^{-3}–10^{-4}, that is, in the case of non-relativistic electrons ($E \sim$ 1 eV) the separation between bands falls into the microwave region. Furthermore, from (11) and (12) the separation is a slowly varying function of the light intensity (I):

$$\Delta E \sim I^{1/4}. \tag{13}$$

Therefore, in most optical experiments available with the present experimental status, the energy separation will similarly fall into the microwave region.

3. Discussion and summary

In intense field electrodynamics (IFE) it is very important to obtain exact or approximate results in analytical form. One of the central problems of IFE is the interaction with free charged particles. The application of such analytical results in high-intensity problems led to the discovery of a series of interesting effects such as, for example, the optically induced level structure called Volkov states (Volkov 1935, Eberly 1968) and intensity-dependent mass shift of free electrons together with intensity-dependent corrections to the Compton scattering (Brown and Kibble 1964, Nikishov and Ritus 1964) etc.

Our aim in the present paper was to give an exact result for the problem of a non-relativistic free electron interacting with a classical electromagnetic plane wave. We have shown that this problem can be reduced to the solution of a Mathieu-type equation. The wavefunction then has the general form as given by equation (8). By considering the stability properties of the solutions, from the eigenvalue equation of Mathieu functions, we deduced that the eigenvalues leading to stable solutions are continuous at low intensities and form bands (optically induced band structure) as the intensity of the external field increases. A similar band structure for relativistic electrons was predicted by Berson and Valdmanis (1973) in the field of two counter-propagating circularly polarised waves, by Cronström and Noga (1977) and by Becker (1977) in a refractive medium. For the non-relativistic problem, however, to our knowledge this is the first case where the possible existence of a band structure is predicted. Experimentally, this could be observed in the following way. A very low-energy monochromatic electron beam populates the lowest band only when a high-intensity optical field is switched on. Therefore a microwave radiation in resonance with the nearest band would be absorbed as a result of interband transitions. Difficulties may, however, arise from stability requirements both for the optical field (frequency and intensity stability) and for the electron beam, and these strict requirements can make observation of the effect more difficult.

Let us now proceed to the discussion of the limiting cases of our theory. If, as usual, in (4) we neglect the f'' term, the remaining first-order equation can be solved exactly by direct integration. The result is

$$f(\nu)=f(0)\exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}\nu\frac{(e/mc)\boldsymbol{A}\boldsymbol{p}-e^2A_0^2/2mc^2}{\hbar\omega(1-v_\parallel/c)}\right). \tag{14}$$

Exactly the same wavefunction was applied in calculations beyond the non-relativistic dipole approximation (Ehlotzky 1978, Nickle 1966). The approximation which leads tc this result is well justified at low anu medium intensities by the smallness of the coefficient of the f'' term in (4) having the numerical value 10^{-5}–10^{-6} at optical frequencies. However, all the interesting new results including band structure come from this term, and as we discussed in the considerations after (5), at extremely high intensities this term may become important. Furthermore, the exponent of the wavefunction (14) contains a periodic function of ν, and if we expand it into Fourier series of $\mathrm{e}^{\pm \mathrm{i}n\nu}$ we see that the Volkov-type level structure is reobtained with energy spacing equal to $\hbar\omega$ and momentum spacing equal to $\hbar\boldsymbol{k}$. Exactly the same result can be obtained from the nr n-relativistic limit of the solution of the Klein–Gordon equation. The connection between (14) and the exact Mathieu solution (8) can be seen most easily by invoking Hill's method (Arscott 1964). The Fourier expansion of the $\mathrm{e}^{\mu z}\phi(z)$ Mathieu function reads

$$\mathrm{e}^{\mu z}\phi(z)=\mathrm{e}^{\mu z}\sum_{r=-\infty}^{\infty} c_{2r}\,\mathrm{e}^{2\mathrm{i}rz}. \tag{15}$$

Introducing this expansion into the Mathieu equation (7), we arrive at the following recurrence relation for the c_{2r} coefficients:

$$\frac{(2r-\mathrm{i}\mu)^2-a}{-\frac{1}{2}q}c_{2r}=c_{2r-2}+c_{2r+2}. \tag{16}$$

At optical frequencies the parameter a is very large ($\sim 10^{10}$) and also $|a/q|\gg 1$. In this case $-\mathrm{i}\mu=\sqrt{a}$, and by introducing the notation $c_{2r}=d_r$, from (16) we obtain

$$2rd_r/(-\tfrac{1}{4}q/\sqrt{a})=d_{r+1}+d_{r+1} \tag{17}$$

and this is just the recurrence relation for Bessel functions of order r (Abramowitz and Stegun 1964), having the solution

$$c_{2r}\cong J_r(u), \qquad u=\frac{eA_0p_\perp}{mc\hbar\omega(1-v_\parallel/c)}. \tag{18}$$

The coefficients in the Fourier expansion of (14) are the same Bessel functions with the same argument. This shows the connection between our result and earlier works and completes the discussion.

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