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LETTER TO THE EDITOR

Classical limit of non-relativistic Compton scattering in external fields†

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Abstract. We investigate the scattering of high-frequency laser radiation by electrons embedded simultaneously in a strong homogeneous magnetic field and an intense microwave field. The microwave field is nearly at resonance with the cyclotron frequency. The scattering of the laser field is treated in the lowest order of perturbation theory, whereas the motion of the electron in the external fields is described exactly. If we consider highly excited Landau states and keep in mind that in the classical limit recoil effects can be neglected, we recover the classical formula which we derived earlier.

In a continuation of previous work (Varró *et al* 1984) we investigate in this Letter the scattering of high-frequency laser radiation by electrons embedded simultaneously in a strong homogeneous magnetic field $\boldsymbol{B}$ and an intense microwave field F. The microwave of frequency ω and wavenumber k will be taken to be nearly at resonance with the cyclotron frequency $\omega_c = eB/Mc$ where M is the electron mass. The scattering of the laser field of frequency Ω, wavevector $\boldsymbol{K}$ and polarisation $\boldsymbol{\varepsilon}$ will be treated in lowest order of perturbation theory, whereas the motion of the electron will be described exactly. We assume ω, $\omega_c \ll \Omega$ in which case the scattering of the laser quanta $\hbar\Omega$ will be dominated by the 'seagull' diagram of figure 1 (see, for example, Sakurai 1975) corresponding to the $\boldsymbol{A}' \cdot \boldsymbol{A}$ part of the non-relativistic interaction Hamiltonian. This approximation simply expresses the fact that for the scattering of laser radiation the electron may be considered instantaneously at rest. The microwave field will be assumed to be circularly polarised and to propagate in the direction of $\boldsymbol{B} = B\boldsymbol{e}_z$ where $\boldsymbol{e}_z$ is a unit vector along the z axis. The low-frequency microwave will be represented in the dipole approximation by the vector potential $\boldsymbol{A}_\omega(t)$ and we shall work throughout in the Coulomb gauge. Consequently, there will be no recoil effects from the microwave.

For the combined strong external magnetic and microwave fields we choose, as in our previous paper, the representation

$$\boldsymbol{A}^{\text{ext}} = \tfrac{1}{2}\boldsymbol{B} \times \boldsymbol{x} + \boldsymbol{A}_\omega = \left(-\tfrac{1}{2}By + \frac{cF}{\omega}\cos\omega t, \tfrac{1}{2}Bx + \frac{cF}{\omega}\sin\omega t, 0\right) \qquad (1)$$

and from elementary quantum electrodynamics we find for the effective interaction

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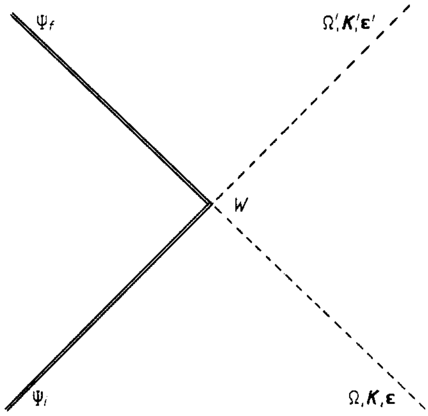

Figure 1. Seagull diagram of non-relativistic Compton scattering of laser light by an electron embedded simultaneously in a constant homogeneous magnetic field B and a microwave field F. The electron may be considered instantaneously at rest so that the other two diagrams of Compton scattering contribute very little in particular for the consideration of the classical limit. The double lines indicate the 'dressed', ingoing and outgoing electron states.

Hamiltonian which follows from the $\boldsymbol{A}' \cdot \boldsymbol{A}$ interaction term (Sakurai 1975)

$$W \equiv r_0 \frac{2\pi\hbar c^2}{(\Omega'\Omega)^{1/2}} (\boldsymbol{\varepsilon}' \cdot \boldsymbol{\varepsilon}) \exp[\mathrm{i}(\boldsymbol{K}-\boldsymbol{K}') \cdot \boldsymbol{x} - \mathrm{i}(\Omega-\Omega')t] \tag{2}$$

where $r_0 = e^2/Mc^2$ and Ω', $\boldsymbol{K}'$ and $\boldsymbol{\varepsilon}'$ characterise the scattered laser field. The scattering processes we want to consider are then determined by the matrix element

$$T_{fi} = \frac{1}{\mathrm{i}\hbar} \int_{-\infty}^{+\infty} \langle \Psi_f | W | \Psi_i \rangle \, \mathrm{d}t \tag{3}$$

where the initial and final electron states $|\Psi\rangle$ are exact solutions of the Schrödinger equation for a charged particle in the external field (1) which means

$$\frac{1}{2M}\left(\hat{\boldsymbol{p}} + \frac{e}{c}\boldsymbol{A}^{\mathrm{ext}}\right)^2 |\Psi\rangle = \mathrm{i}\hbar\partial_t|\Psi\rangle \tag{4a}$$

and this equation can be written in the form

$$\frac{1}{2M}\left[\left(\hat{p} + \frac{eF}{\omega}\cos\omega t\right)^2 + \left(M\omega_c\hat{q} + \frac{eF}{\omega}\sin\omega t\right)^2 + \hat{p}_z^2\right]|\Psi\rangle = \mathrm{i}\hbar\partial_t|\Psi\rangle \tag{4b}$$

where

$$\hat{\boldsymbol{p}} \equiv \hat{\boldsymbol{p}}_x - \tfrac{1}{2}M\omega_c\hat{y} \qquad M\omega_c\hat{\boldsymbol{q}} \equiv \hat{\boldsymbol{p}}_y + \tfrac{1}{2}M\omega_c\hat{x} \qquad [\hat{\boldsymbol{q}}, \hat{\boldsymbol{p}}] = \mathrm{i}\hbar \tag{4c}$$

and ω_c is the cyclotron frequency defined above. In our previous paper (Varró *et al* 1984) we have shown that the solution of (4b) can be written in the form

$$|\Psi\rangle = |p_z\rangle D|\Phi\rangle \tag{5}$$

where

$$\langle z|p_z\rangle = (2\pi\hbar)^{-1/2}\exp\left(\frac{\mathrm{i}}{\hbar}[p_z z - (p_z^2/2M)t]\right) \tag{5a}$$

is a plane wave with momentum p_z and

$$D=\exp\left[\mathrm{i}\left(\frac{\mu}{\hbar k}\right)\left(\frac{\omega}{\Delta\omega}\right)(\hat{p}\sin\omega t-M\omega_c\hat{q}\cos\omega t)\right]. \tag{5b}$$

This displacement operator D eliminates from (4b) the microwave $\boldsymbol{A}_\omega$ so that $|\Phi\rangle$ of equation (5) satisfies the Schrödinger equation

$$\left(\frac{\hat{p}^2}{2M}+\tfrac{1}{2}M\omega_c^2\hat{q}^2\right)|\Phi\rangle=\mathrm{i}\hbar\partial_t|\Phi\rangle \tag{5c}$$

which describes the motion of an electron in the magnetic field $\boldsymbol{B}$. In (5b) we have introduced $k=\omega/c$ and the displacement from resonance $\Delta\omega=\omega-\omega_c\neq 0$. The dimensionless parameter $\mu\equiv(eF/M\omega c)$ characterises the intensity of the microwave.

The stationary-state solutions of (5c) are the well known Landau states (Landau and Lifshitz 1977) which have the following representation in cylindrical coordinates ρ, φ

$$\begin{aligned}\langle\rho\varphi|\Phi\rangle&=\left(\frac{\gamma}{\pi}\right)^{1/2}\left(\frac{l!}{(|m|+l)!}\right)^{1/2}\exp(\mathrm{i}m\varphi)\exp(-\xi^2/2)\xi^{|m|/2}L_l^{|m|}(\xi)\exp(-\mathrm{i}E_{lm}t/\hbar)\\&=\phi_{lm}\exp(-\mathrm{i}E_{lm}t/\hbar)\end{aligned} \tag{6}$$

with the following parameters

$$\xi=\gamma\rho^2 \qquad \gamma=(eB/2\hbar c)=(M\omega_c/2\hbar)$$

and the $L_l^{|m|}$ denote associated Laguerre polynomials (Gradshteyn and Ryzhik 1980). The corresponding energy eigenvalues are then given by

$$E_{lm}=\hbar\omega_c[l+\tfrac{1}{2}(m+|m|+1)] \qquad m=0,\pm1,\pm2,\ldots \qquad l=0,1,2,\ldots. \tag{6a}$$

By means of the definitions given in (4c) for $\hat{p}$ and $\hat{q}$ and by applying the Baker–Haussdorff formula for two Bose operators A and B

$$\mathrm{e}^{A+B}=\mathrm{e}^A\mathrm{e}^B\mathrm{e}^{-[A,B]/2} \qquad [A,[A,B]]=[B,[A,B]]=0$$

we can easily show that the displacement operator (5b) can be rewritten in the form

$$\begin{aligned}D=&\exp\left[-\frac{\mathrm{i}}{2}M\omega_c\left(\frac{\mu}{\hbar k}\right)\left(\frac{\omega}{\Delta\omega}\right)(x\cos\omega t+y\sin\omega t)\right]\\&\times\exp\left[\left(\frac{\mu}{k}\right)\left(\frac{\omega}{\Delta\omega}\right)(\sin\omega t\partial_x-\cos\omega t\partial_y)\right].\end{aligned} \tag{7}$$

By taking into account the formulae (2), (5), (6) and (7) we can write the matrix element (3) in the form

$$\begin{aligned}T_{fi}=&r_0\frac{2\omega\hbar c^2}{(\Omega'\Omega)^{1/2}}(\boldsymbol{\varepsilon}'\cdot\boldsymbol{\varepsilon})\delta(P_z-P_z')\langle\phi_{l'm'}|\exp(\mathrm{i}\boldsymbol{Q}^\perp\cdot\boldsymbol{x})|\phi_{lm}\rangle\\&\times\frac{1}{\mathrm{i}\hbar}\int_{-\infty}^{+\infty}\mathrm{d}t\exp\left(-\frac{\mathrm{i}}{\hbar}(E-E')t\right)\exp(-\mathrm{i}\boldsymbol{Q}^\perp\cdot\boldsymbol{x}_c(t))\end{aligned} \tag{8}$$

where $\boldsymbol{x}_c(t)=(\mu/k)(\omega/\Delta\omega)(\sin\omega t,-\cos\omega t,0)$ is just the transverse part of the trajectory of a classical electron moving in the two external fields. Moreover we have introduced in (8) the total energy $E\equiv\hbar\Omega+E_{lm}+p_z^2/2M$ before, and correspondingly

E' after the scattering event and we have denoted the z component of the total momentum of the system by $P_z \equiv p_z + \hbar K_z$ and P'_z respectively for the initial and final states of the system. The transverse part of the momentum transferred during the scattering process is given by $\boldsymbol{Q}^\perp \equiv (K_x - K'_x, K_y - K'_y, 0)$.

The integration over time can be performed in (8) after we have expanded $\exp(-\mathrm{i}\boldsymbol{Q}^\perp \cdot \boldsymbol{x}_\mathrm{c}(t))$ into a Fourier series which just defines the ordinary Bessel functions J_n of the order n on account of the Jacobi–Anger formula (Erdély 1953). We thus obtain an infinite sum of matrix elements for incoherent scattering processes of different frequencies $\Omega' = \Omega_{n\nu}$ of the emitted radiation

$$T_{fi} = \sum_{n=-\infty}^{+\infty} T_{fi}^{(n)} \qquad T_{fi}^{(n)} \equiv -2\pi\mathrm{i}\delta(P_z - P'_z)\delta(E + n\hbar\omega - E')t_{fi}^{(n)} \tag{9}$$

where

$$t_{fi}^{(n)} \equiv r_0 \frac{2\pi\hbar c^2}{(\Omega'\Omega)^{1/2}} (\boldsymbol{\varepsilon}' \cdot \boldsymbol{\varepsilon}) J_n[\mu(\omega/\Delta\omega)Q^\perp/k] \exp(\mathrm{i}n\chi)\langle\phi_{l'm'}|\exp(\mathrm{i}\boldsymbol{Q}^\perp \cdot \boldsymbol{x})|\phi_{lm}\rangle \tag{9a}$$

and the phase χ is defined by $\sin\chi \equiv (Q_y/Q^\perp)$ whereas $Q^\perp \equiv |\boldsymbol{Q}^\perp| = (Q_x^2 + Q_y^2)^{1/2}$.

If the longitudinal momentum component p_z of the electron is initially zero then the relative frequency change (and photon momentum change) during the scattering process on account of the longitudinal recoil will be of the order of magnitude $(\hbar\Omega/Mc^2)$ which is comparatively small for optical photons.

Consequently, we obtain for the frequencies of scattered radiation

$$\Omega' = \Omega_{n\nu} \equiv \Omega + n\omega + \nu\omega_\mathrm{c} \tag{10}$$

where $\nu \equiv l - l' + \frac{1}{2}(m + |m| - m' - |m'|)$ is the change in the excitation indices of the Landau states. Moreover, n, $\nu > 0$ correspond to induced absorption of microwave and synchrotron quanta whereas n, $\nu < 0$ correspond to stimulated emission. The longitudinal momentum change may become important if it is of compariable magnitude with the transverse momentum transfer (as in the case of backscattering). We shall not, however, consider this case here.

From (9*a*) we can now evaluate the differential cross sections for the various scattering processes by means of standard procedures to obtain

$$\frac{\mathrm{d}\sigma_{n\nu}}{\mathrm{d}\Omega_l} = r_0^2(\boldsymbol{\varepsilon}' \cdot \boldsymbol{\varepsilon})^2 \left(\frac{\Omega_{n\nu}}{\Omega}\right) J_n^2[\mu(\omega/\Delta\omega)Q^\perp/k] |\langle\phi_{l'm'}|\exp(\mathrm{i}\boldsymbol{Q}^\perp \cdot \boldsymbol{x})|\phi_{lm}\rangle|^2 \tag{11}$$

where Ω_l refers to the angular element of radiation emission and we should recognise that $Q^\perp$ also depends on n, ν via $\Omega_{n\nu}$ of (10). In order to evaluate for (11) a still more explicit expression, we have to calculate the matrix elements of $\exp(\mathrm{i}\boldsymbol{Q}^\perp \cdot \boldsymbol{x})$ between different Landau states (6). To this end we first observe that the angular integration over φ can be easily performed if we again use the Fourier expansion of the above exponential into Bessel functions to obtain the following formula for a remaining integration over ξ

$$\begin{aligned} &\langle\phi_{l'm'}|\exp(\mathrm{i}\boldsymbol{Q}^\perp \cdot \boldsymbol{x})|\phi_{lm}\rangle \\ &\quad = \left(\frac{l'!}{(|m'| + l')!}\,\frac{l!}{(|m| + l)!}\right)^{1/2} \exp[\mathrm{i}(m' - m)(\chi + \pi/2)] \\ &\qquad \times \int_0^\infty \mathrm{d}\xi\, \mathrm{e}^{-\xi}\xi^{(|m'|+|m|)/2} L_{l'}^{|m'|}(\xi) L_l^{|m|}(\xi) \dot{J}_{m'-m}[Q^\perp(\xi/\gamma)^{1/2}]. \end{aligned} \tag{12}$$

The general evaluation of the last integral causes, however, considerable difficulties.

Since in this investigation we are particularly interested in the correspondence between the quantum and classical descriptions of the above scattering processes, we shall consider in the following transitions between highly excited Landau states. Those Landau states with negative values of angular momentum ($m < 0$) are typical quantum states, because the classical cyclotron gyration of the electron can never have negative angular momentum values in the magnetic field configuration considered. Therefore, for the classical limit we shall take m, $m' > 0$. Moreover, if we consider a coherent superposition of highly excited Landau states which correspond to a gyration along a classical trajectory of radius $\rho_0 \gg \gamma^{-1/2}$, then we see that for the dominant term in such a superposition the relation $m = \gamma\rho_0^2$ holds (see Varró *et al* 1984 where a coherent superposition of states ϕ_{0m} has been discussed). In addition it can be generally shown (Varró 1984) that for an arbitrary coherent superposition of Landau states, the contribution of those states ϕ_{lm} with $l \neq 0$ determine in the (x, y) plane the location of the centre of the circle along which the packet of Landau states gyrates. Now, from the point of view of the scattering of light the average position of the scatterer (and the location of its centre of gyration) is completely immaterial, if only it does not change considerably during the scattering process. Furthermore, we have assumed at the very beginning that $\Omega \gg \omega_c$ and therefore the incident laser light cannot induce resonant transitions between the Landau states, which of course also requires $\Omega_{n\nu} \gg \omega_c$. The latter condition means that very high orders of non-linearities are not permitted to occur which is fortunately taken care of by the properties of the Bessel functions. Consequently, the laser light can also not disintegrate the packet of highly excited Landau states and therefore the momentum transferred during the scattering process acts on the wavepackets as a whole and in our case such recoil effects are negligible. This then means that the location of the centre of gyration will not be considerably changed during our scattering process and we may as well keep it fixed and assume that it is at the origin of the (x, y) plane for which $l = 0$. In passing we remark that coherent superpositions of states ϕ_{0m} correspond for high excitations to classical particle gyration around the origin.

Consequently, for our considerations of the classical limit we set l, $l' = 0$. Then we have $L_0^{|m|} = 1$ for any $|m|$ and the integral of (12) becomes much simpler (Gradshteyn and Ryzhik 1980) to yield

$$\langle\phi_{0,m'}|\exp(\mathrm{i}\boldsymbol{Q}^{\perp}\cdot\boldsymbol{x})|\phi_{0m}\rangle = \left(\frac{m!}{m'!}\right)^{1/2} \exp\left(-\frac{Q^{\perp 2}}{4\gamma}\right)\left(\frac{Q^{\perp 2}}{4\gamma}\right)^{(m'-m)/2} L_m^{m'-m}\left(\frac{Q^{\perp 2}}{4\gamma}\right) \qquad m' > m \qquad (12a)$$

and a similar expression holds for $m' < m$. For considering the classical limit, we assume that m, $m' \to \infty$ while $m' - m = \nu$ is kept fixed. Moreover, we have discussed before that for a packet of states $\phi_{0\mu}$ the dominant term $\mu = m$ is characterised by the relation $\gamma\rho_c^2 = m$ and this is also true for the maximum of the probability distribution for ϕ_{0m}. We therefore make in (12a) the replacement $1/\gamma = \rho_c^2/m$ and while letting $m \to \infty$ we take into account the following asymptotic formula for the associated Laguerre polynomials (Gradshteyn and Ryzhik 1980)

$$\lim_{m\to\infty}\left[\frac{1}{m^{\nu}} L_m^{\nu}\left(\frac{x}{m}\right)\right] = x^{-\nu/2} J_\nu(2\sqrt{x})$$

which consequently yields by means of (11) and (12a) the following cross section formula

$$\frac{\mathrm{d}\sigma_{n\nu}}{\mathrm{d}\Omega_l}=r_0^2(\boldsymbol{\varepsilon}'\cdot\boldsymbol{\varepsilon})^2\left(\frac{\Omega_{n\nu}}{\Omega}\right)J_n^2\left[\frac{\mu}{k}\left(\frac{\omega}{\Delta\omega}\right)Q_{n\nu}^{\perp}\right]J_\nu^2(\rho_c Q_{n\nu}^{\perp}) \tag{13}$$

which after summing and averaging over the directions of polarisation of the laser light exactly agrees with our result of calculations using classical electrodynamics (Varró *et al* 1984). Since in classical electrodynamics quantum mechanical recoil effects are unknown and usually radiation damping is also neglected, we expect no recoil effects to occur in a classical calculation of our radiation processes. Therefore, in the light of our approach to the classical limit from the quantum mechanical result the assumption of a fixed centre of gyration with $l=0$ appears to be a very natural one. But further clarifying work has still to be done in this domain. In closing we should like to refer to the recent work of Lieu *et al* (1983) and Lieu (1984) on closely related problems.

References

Erdély A (ed) 1953 *Higher Transcendental Functions* vol 2 (New York: McGraw-Hill)
Gradshteyn I S and Ryzhik I M 1980 *Tables of Integrals, Series and Products* (New York: Academic)
Landau L D and Lifshitz E M 1977 *Quantum Mechanics* (London: Pergamon)
Lieu R 1984 *J. Phys. A: Math. Gen.* **17** L223
Lieu R, Leahy D A and Evans A J 1983 *J. Phys. A: Math. Gen.* **16** L669
Sakurai J J 1975 *Advanced Quantum Mechanics* (Reading, MA: Addison-Wesley).
Varró S 1984 *J. Phys. A: Math. Gen.* **17** 1631–8
Varró S, Ehlotzky F and Bergou J 1984 *J. Phys. B: At. Mol. Phys.* **17** 483