Classical limit of Compton scattering and electron scattering in external fields†

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Abstract. In this paper we investigate simultaneously two non-relativistic scattering problems from a unifying point of view, namely: (a) scattering of light by a free electron; and (b) scattering of electrons by a screened Coulomb potential. In both cases the electrons involved in the scattering process are embedded in an intense microwave field $F(t)$ and in a strong homogeneous magnetic field B. Therefore these particles have to be described by exact field-dressed states. On the other hand, the basic scattering processes (a) and (b) will be treated to lowest order of perturbation theory. In particular we shall investigate the non-linear processes induced by the strong external fields in the classical limit of highly excited quantum states. In this case we shall be able to derive compact analytic expressions for the transition amplitudes and cross sections.

1. Introduction

The investigation of laser-induced and laser-assisted scattering processes has been of great interest during the past twenty years (Ehlotzky 1985). Inverse bremsstrahlung has been considered as one of the important processes involved in the heating of fusion plasmas by laser radiation (Seely 1974a, b, Ferrante 1985). The problem of electron motion and scattering in a strong radiation pulse and in a homogeneous magnetic field involves some additional complications concerning the proper formulation of the boundary conditions (Faisal 1982, Zarcone *et al* 1983).

In our recent work, we have therefore decided to first investigate the problem of non-relativistic Compton scattering in the combined strong external fields mentioned above, since this problem apparently allows for a corresponding classical treatment (Varró *et al* 1984, Varró and Ehlotzky 1984). In the present paper we shall continue our research by demonstrating the existence of a close analogy between Compton scattering and electron scattering in the combined external fields, in particular with regard to the evaluation of the matrix elements of these processes and the consideration of the proper classical limits.

We shall study in the following sections the scattering of light by a free electron of mass M and charge $-e$ which is simultaneously embedded in an intense microwave field $F(t)$ and in a strong homogeneous magnetic field B. At the same time we shall consider the scattering of the electrons by a screened Coulomb potential in the presence of the same external field configuration. The particular choice of the external fields

will be specified in § 2. There we shall also write down the interaction Hamiltonians for Compton scattering and electron scattering respectively. Describing the ingoing and outgoing electrons by exact field-dressed states we shall moreover write down in this section the corresponding scattering matrix elements to lowest order of perturbation theory. The next section will be devoted to the discussion and the detailed evaluation of these matrix elements and, in particular, the similarities between Compton scattering and electron scattering will be pointed out. In § 4 we shall finally calculate the differential scattering rates and cross sections of these two processes and we shall consider in particular the non-linear effects induced by the external fields in the classical limit of highly excited quantum states. We shall conclude our work with a short discussion of our main results in § 5.

2. The scattering matrix elements

For the following considerations we shall assume that the microwave is right-hand circularly polarised and that the wave propagates in the direction of the magnetic field $B = B\hat{e}_z$ where $\hat{e}_z$ is a unit vector along the z axis. The low frequency of the microwave permits us to represent this wave in the dipole approximation by the vector potential $A_\omega(t)$. For the homogeneous magnetic field we shall adopt the vector potential in the symmetric gauge form $\frac{1}{2}B \times x$. Thus we shall represent the combined external fields by the vector potential

$$A^{\text{ext}} = \frac{1}{2}B \times x + A_\omega(t)$$
$$= [-\frac{1}{2}By + (eF/\omega)\cos \omega t, \frac{1}{2}Bx + (eF/\omega)\sin \omega t, 0]. \tag{1}$$

In the following we shall describe the two non-relativistic scattering processes to the lowest order of perturbation theory. The basic set of electron states will be given by the exact solutions of the Schrödinger equation for the external fields equation (1). This means that we perform perturbation theory in the Furry picture.

2.1. Compton scattering

We begin our investigations with the consideration of Compton scattering since this turns out to be the simpler process. As in our foregoing work (Varró and Ehlotzky 1984) we shall assume that the frequency ω of the microwave and the cyclotron frequency $\omega_c \equiv eB/Mc$ are much smaller than the frequency Ω of the scattered light. In this case the dominant contribution to the amplitude of Compton scattering is determined by the 'seagull diagram' of figure 1 (see, for example, Sakurai 1975). This diagram corresponds to the $A' \cdot A$ part of the non-relativistic interaction Hamiltonian and its contribution can be represented by the following effective interaction potential (Varró and Ehlotzky 1984)

$$W \equiv r_0 \frac{2\pi\hbar c^2}{(\Omega'\Omega)^{1/2}} (\varepsilon' \cdot \varepsilon) \exp[i(K - K')x - i(\Omega - \Omega')t]. \tag{2}$$

In this expression the frequencies, wavevectors and polarisations of the ingoing and scattered Compton light are denoted by Ω, K, ε and Ω', K', ε' respectively and r_0 is the classical electron radius. In figure 1 the double lines represent the exact field-dressed states of the ingoing and outgoing electron.
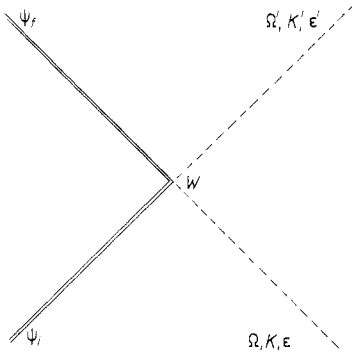
Figure 1. 'Seagull diagram' of non-relativistic Compton scattering of laser light by an electron embedded simultaneously in a constant homogeneous magnetic field B and a microwave field F. The electron may be considered instantaneously at rest so that the other two diagrams of Compton scattering contribute very little in particular for the consideration of the classical limit. The double lines indicate the 'dressed' ingoing and outgoing electron states.
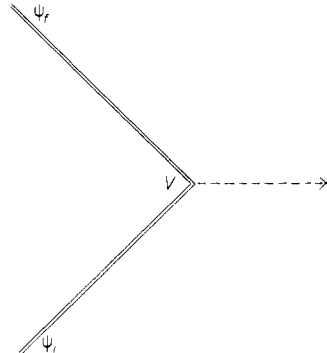
Figure 2. Diagram of non-relativistic electron scattering by a screened Coulomb potential V in lowest order Born approximation. The double lines indicate the states of the ingoing and outgoing electron which are 'dressed' by the combined action of the external fields B and F.

2.2. Electron scattering

Turning now to the process of electron scattering we shall consider as interaction Hamiltonian a screened Coulomb potential of the form $V(r) = (A/r)\exp(-\alpha r)$ where A is an amplitude to be specified later and α^{-1} is the screening length. It will be convenient for our discussions later on to decompose $V(r)$ into its Fourier components. This yields a superposition of static plane waves of wavevectors $k \equiv (k_\perp, k_z)$ and we obtain

$$V = [4\pi A/(2\pi)^3] \int d^2 k_\perp \, dk_z \frac{\exp[i(k_\perp \cdot x_\perp + k_z z)]}{\alpha^2 + k_\perp^2 + k_z^2}. \tag{3}$$

Here we have introduced $k_\perp \equiv (k_x, k_y)$ and $x_\perp \equiv (x, y)$ as two-dimensional vectors in a plane perpendicular to $\hat{e}_z$. The diagram which corresponds to the lowest order Born approximation of the electron scattering amplitude is shown in figure 2 where the double lines again indicate the field-dressed particle states.

The scattering processes (a) and (b) which are represented by figures 1 and 2 respectively are then described by the matrix elements

$$T_{fi} = (i\hbar)^{-1} \int_{-\infty}^{+\infty} dt \langle \psi_f | H' | \psi_i \rangle \tag{4}$$

where in the case of Compton scattering the effective interaction Hamiltonian $H' \equiv W$ as given by equation (2) and for electron scattering we have $H' = V$ which is represented by equation (3).

With reference to perturbation theory in the Furry picture the initial and final states $|\psi\rangle$ in equation (4) are solutions of the Schrödinger equation for an electron in the external fields, equation (1), and therefore

$$(2M)^{-1}[\hat{p} + (e/c)A^{\text{ext}}]^2 |\psi\rangle = i\hbar \partial_t |\psi\rangle. \tag{5a}$$

The exact solutions of this equation (Varró and Ehlotzky 1984) can be labelled by the three quantum numbers l, m and p_z. The continuous parameter p_z represents the linear momentum of the free electron motion along the z direction with energy $p_z^2/2M$. l and m are the discrete radial and magnetic quantum numbers respectively which characterise the simultaneous transverse motion (perpendicular to $\hat{e}_z$) of the electron. These solutions $|\psi\rangle$ of equation (5a) can be written down in the following form

$$|\psi\rangle = |p_z\rangle D|\Phi\rangle \tag{5b}$$

where

$$\langle z|p_z\rangle = (2\pi\hbar)^{-1/2} \exp\{(i/\hbar)[p_z z - (p_z^2/2M)t]\} \tag{5c}$$

and

$$D = \exp[i(a/\hbar)(\hat{p}\sin\omega t - M\omega_c\hat{q}\cos\omega t)]$$

$$\hat{p} = \hat{p}_x - \tfrac{1}{2}M\omega_c\hat{y} \qquad M\omega_c\hat{q} = \hat{p}_y + \tfrac{1}{2}M\omega_c\hat{x} \qquad [\hat{q}, \hat{p}] = i\hbar. \tag{5d}$$

In the unitary operator D, given by equation (5d), the amplitude a of the classical electron motion in the microwave appears as the essential parameter. This is defined by (Varró *et al* 1984)

$$a = (eF/M\omega c)(\omega/\Delta\omega)\lambda \qquad \Delta\omega = \omega - \omega_c \neq 0 \qquad \lambda = c/\omega. \tag{6}$$

The modulus $|a|$ of this amplitude is essentially determined by the dimensionless intensity parameter of the microwave $\mu^2 = (eF/M\omega c)^2$ and the detuning $|\Delta\omega| = |\omega - \omega_c|$ between the microwave and cyclotron frequencies.

According to equation (5b) the solutions $|\psi\rangle$ are found from the solutions $|\Phi\rangle$ by means of the unitary transformation equation (5d). The states $|\Phi\rangle$, however, are the ordinary stationary Landau states which are given in cylindrical coordinates (ρ, φ) by (Landau and Lifshitz 1977)

$$\langle\rho, \varphi|\Phi\rangle = (\gamma/\pi)^{1/2}\left(\frac{l!}{(l+|m|)!}\right)^{1/2} \exp(im\varphi) \exp(-\xi^2/2)\xi^{|m|/2}L_l^{|m|}(\xi) \exp(-iE_{lm}t/\hbar)$$

$$= \Phi_{lm} \exp(-iE_{lm}t/\hbar) \tag{7a}$$

where the $L_l^{|m|}$ are associated Laguerre polynomials (Gradshteyn and Ryzhik 1980) and

$$\xi = \gamma\rho^2 \qquad \gamma = (eB/2\hbar c) = (M\omega_c/2\hbar). \tag{7b}$$

Correspondingly, the discrete energy eigenvalues E_{lm} of the transverse electron motion (Landau levels) are known to be

$$E_{lm} = \hbar\omega_c[l + \tfrac{1}{2}(m + |m| + 1)]$$

$$l = 0, 1, 2, \ldots \qquad m = 0, \pm 1, \pm 2, \ldots. \tag{7c}$$

2.3. Matrix elements of Compton scattering

For the simpler case of Compton scattering we have shown in our previous paper (Varró and Ehlotzky 1984) that the matrix element equation (4) can be decomposed into an infinite sum of matrix elements for various incoherent non-linear scattering

processes of different frequencies $\Omega' = \Omega_{n\nu}$ of the emitted Compton radiation, namely

$$T_{fi} = \sum_{n=-\infty}^{+\infty} T_{fi}^{(n)}$$

$$T_{fi}^{(n)} = -2\pi i \delta(P_z - P'_z)\delta(\bar{E} + n\hbar\omega - \bar{E}')t_{fi}^{(n)} \tag{8}$$

where

$$t_{fi}^{(n)} = r_0 \frac{2\pi\hbar c^2}{(\Omega'\Omega)^{1/2}}(\boldsymbol{\varepsilon}' \cdot \boldsymbol{\varepsilon})J_n(Q_\perp a) \exp[i(\theta + \pi)n]\langle l'm'| \exp(i\boldsymbol{Q}_\perp \cdot \hat{\boldsymbol{x}}_\perp)|lm\rangle. \tag{8a}$$

In equation (8) we have introduced the parameters of total energy $\bar{E} = \hbar\Omega + E_{lm} + p_z^2/2M$ before and correspondingly $\bar{E}'$ after the scattering event and we have denoted the z component of the total linear momentum of the system by $P_z = p_z + \hbar K_z$ and $P'_z = p'_z + \hbar K'_z$ for the initial and final states respectively. Moreover, there appear in equation (8a) the ordinary Bessel functions J_n of the order n and the transverse part of the momentum transfer $\hbar \boldsymbol{Q}_\perp$ is determined by

$$\boldsymbol{Q}_\perp = (K_x - K'_x, K_y - K'_y) \equiv Q_\perp(\cos\theta, \sin\theta). \tag{8b}$$

For the transverse part of the position operator we have written in equation (8a), $\hat{\boldsymbol{x}}_\perp \equiv (\hat{x}, \hat{y})$. The transition matrix elements $t_{fi}^{(n)}$ of equation (8a) are essentially determined by the form factors $\langle l'm'| \exp(i\boldsymbol{Q}_\perp \cdot \hat{\boldsymbol{x}}_\perp)|lm\rangle$ evaluated between two different Landau states $|lm\rangle$, equation (7a).

If the longitudinal component p_z of the electron momentum is zero initially, the frequencies of scattered radiation can be well approximated by the formula

$$\Omega' = \Omega_{n\nu} \equiv \Omega + n\omega - \nu\omega_c$$

$$\nu = l' - l + \tfrac{1}{2}(m' + |m'| - m - |m|). \tag{9}$$

2.4. Matrix elements of electron scattering

We now turn to the problem of electron scattering in the presence of the external field configuration equation (1). The evaluation of the corresponding matrix element, equation (4), with H' given by equation (3) can be carried out along the same lines as for Compton scattering and we again obtain an infinite sum of matrix elements for the different incoherent scattering processes. This yields

$$T_{fi}^{(sc)} = -2\pi i \sum_{n=-\infty}^{+\infty} \delta(E + n\hbar\omega - E')M_{fi}^{(n)} \tag{10}$$

with

$$M_{fi}^{(n)} \equiv \frac{4\pi A}{(2\pi)^3\hbar} \int \frac{d^2k_\perp}{\alpha^2 + q^2 + k_\perp^2} J_n(k_\perp a) \exp[in(\chi + \pi)]\langle l'm'| \exp(i\boldsymbol{k}_\perp \cdot \hat{\boldsymbol{x}}_\perp)|lm\rangle. \tag{10a}$$

In the last equation the longitudinal momentum transfer $\hbar q = p'_z - p_z$ has been introduced and for the integration over the transverse momenta $\hbar \boldsymbol{k}_\perp$ we have defined polar coordinates by putting $\boldsymbol{k}_\perp \equiv k_\perp(\cos\chi, \sin\chi)$. As in the case of Compton scattering we have also introduced in equation (10) the total energy $E = E_{lm} + p_z^2/2M$ of the electron before and correspondingly E' after the scattering event. All other quantities appearing in equation (10a) have been defined before.

By considering the matrix elements of equations (8a) and (10a) we realise that apart from the $k_\perp$ integration in equation (10a) the matrix elements of Compton scattering and electron scattering have a very similar structure and this circumstance permits us to carry out a parallel analysis of these two scattering processes. However, it becomes immediately clear that the matrix elements of Compton scattering can be more easily handled than those of electron scattering.

3. Evaluation of the matrix elements

In order to analyse in detail the structure of the scattering amplitudes we must first explicitly evaluate the matrix elements of the form $\langle l'm'| \exp(\mathrm{i}k_\perp \cdot \hat{x}_\perp)|lm\rangle$ which appear in equations (8a) and (10a). By taking into account the representation of the Landau states equations (7a, b) we obtain after integration over the azimuth φ

$$\langle l'm'| \exp(\mathrm{i}k_\perp \cdot \hat{x}_\perp)|lm\rangle = \mathrm{i}^\mu \exp(-\mathrm{i}\mu\chi)\left(\frac{l'!}{(l'+|m'|)!} \frac{l!}{(l+|m|)!}\right)^{1/2} I(k_\perp) \qquad (11a)$$

where

$$I(k_\perp) \equiv 2\int_0^\infty \mathrm{d}x \, \exp(-x^2)x^{|m'|+|m|+1}L_{l'}^{|m'|}(x^2)L_l^{|m|}(x^2)J_n(k_\perp bx) \qquad (11b)$$

with

$$\mu = m' - m \qquad b = \gamma^{-1/2}. \qquad (11c)$$

In the following we shall not consider all possible combinations of the quantum numbers l, l' and m, m' but we shall concentrate on the special choice m', $m > 0$ and $m' - m = \mu > 0$. This choice is not essential for the analysis to be carried out below but it is of particular interest for the consideration of the classical limit which will be investigated later on.

As it turns out, it is convenient for the treatment of Compton scattering as well as of electron scattering to first evaluate an explicit expression for the integral $I(k_\perp)$ as defined by equation (11b). By means of the Rodrigues formula which defines the associated Laguerre polynomials we can easily prove that

$$L_{l'}^m(x^2) = [(l+m)!/l!](-1)^m x^{-2m}L_{l+m}^{-m}(x^2) \qquad (12a)$$

and therefore $I(k_\perp)$ can be brought to the form

$$I(k_\perp) = 2[(l+m)!/l!](-1)^m \int_0^\infty \mathrm{d}x \, \exp(-x^2)x^{\mu+1}L_{l'}^{m'}(x^2)L_{l+m}^{-m}(x^2)J_\mu(k_\perp bx). \qquad (12b)$$

Although an explicit analytic expression for this integral can be found in Gradshteyn and Ryzhik (1980), one can show that the formula (7.422.2) presented by these authors is incorrect. This can be easily demonstrated by considering the integral equation (12b) as the Hankel transform of the function $\exp(-x^2)x^{\mu+1/2}L_{l'}^{m'}(x^2)L_{l+m}^{-m}(x^2)$. If we use the above formula of Gradshteyn and Ryzhik then the inverse transform of the image function does not coincide with the original function as it should do. Therefore the transformation formula of Gradshteyn and Ryzhik mentioned above cannot be the proper one. In the course of our calculations we have found the correct Hankel

transformation according to which $I(k_\perp)$ is given by

$$I(k_\perp) = (-1)^{l'+l} [(l+m)!/l!] \beta^\mu L_{l'}^{-\lambda}(\beta^2) L_{l+m}^{\mu+\lambda}(\beta^2) \exp(-\beta^2) \qquad (13)$$

where

$$\beta = k_\perp b/2 \qquad \lambda = l' - l. \qquad (13a)$$

These results will permit us to investigate Compton scattering and electron scattering below in more detail.

3.1. Compton scattering

By means of the explicit expression, equations (13) and (13a), for $I(k_\perp)$ we are able to write down the amplitudes for Compton scattering $t_{fi}^{(n)}$ in closed form on account of equations (8a) and (11a, b). These amplitudes can then be used to evaluate by standard methods the corresponding transition probabilities and differential cross sections. After summation and averaging over the polarisations of the scattered light these calculations yield the following cross section formulae

$$d\sigma_{n\nu} = d\sigma_{\text{Th}}(\Omega_{n\nu}/\Omega) J_n^2(Q_\perp a) I_{l,\lambda}^2(Q_\perp) I_{l+m,\nu}^2(Q_\perp) \qquad (14)$$

where $d\sigma_{\text{Th}}$ denotes the differential cross section of ordinary Thomson scattering. Moreover we have introduced the notation

$$I_{l,\lambda}(Q_\perp) \equiv [l!/(l+\lambda)!]^{1/2} L_l^\lambda(\beta^2) \beta^\lambda \exp(-\beta^2/2). \qquad (14a)$$

According to equation (13a) we have to put here $\beta = Q_\perp b/2$ where $Q_\perp$ represents the modulus of the transverse part of the momentum transfer which has been introduced in equation (8b). $Q_\perp$ depends weakly on n and $\nu = -(\mu + \lambda)$ via the scattered frequencies $\Omega_{n\nu}$ defined by equation (9).

3.2. Electron scattering

The treatment of the electron scattering problem is much more complicated. If we insert the matrix elements, equations (11a, b), into equation (10a) and then perform the integration over χ we obtain for the matrix elements $M_{fi}^{(n)}$ of electron scattering the expression

$$M_{fi}^{(n)} = \delta_{n\mu} [4\pi A/(2\pi)^2 \hbar] (-i)^\mu \left(\frac{l'!}{(l'+|m'|)!} \frac{l!}{(l+|m|)!} \right)^{1/2}$$

$$\times 2 \int_0^\infty k_\perp \, dk_\perp \frac{J_\mu(k_\perp a)}{\alpha^2 + q^2 + k_\perp^2} \int_0^\infty dx \exp(-x^2) x^{m'+m+1}$$

$$\times L_{l'}^{m'}(x^2) L_l^m(x^2) J_\mu(k_\perp bx). \qquad (15)$$

The Kroenecker factor $\delta_{n\nu}$ at the beginning of this formula expresses the conservation of angular momentum during electron scattering and this is a direct consequence of the cylindrical symmetry of the scattering potential equation (3). Since the angular momentum of the microwave photons is $\hbar$ we obtain on account of the emission (or absorption) of n such photons a corresponding increase (or decrease) of the electron's angular momentum by the amount $n\hbar = \hbar(m' - m) = \hbar\mu$, (see equation (11a)). This conclusion does not hold for Compton scattering since in that case the effective interaction potential equation (2) has no cylindrical symmetry.

By means of equations (15), (12b), (13) and (14a) we can now write down the transition amplitudes in the form

$$M_{fi}^{(n)} = \delta_{n\mu}[4\pi A/(2\pi)^2\hbar](-1)^\mu \int_0^\infty k_\perp \, dk_\perp \frac{J_\mu(k_\perp a)}{\alpha^2 + q^2 + k_\perp^2} I_{l,\lambda}(k_\perp) I_{l+m,\lambda}(k_\perp). \qquad (16)$$

It appears very unlikely, if not impossible, to find in the general case of this exact expression for $M_{fi}^{(n)}$ a simple and closed analytic formula as we were able to derive in § 3.1 for Compton scattering. In the latter case there was no additional integration to be carried out in equation (16). However, the integral of equation (16) can be evaluated if we treat the transverse motion of the electron in the classical limit. This will be shown in the next section.

Before considering the classical limit, however, we should like to indicate another approach to the evaluation of the double integral on the right-hand side of equation (15). Here we first do the integration with respect to $k_\perp$. We obtain (Gradshteyn and Ryzhik 1980, formula 6.541.1)

$$M_{fi}^{(n)} = \delta_{n,\mu}[4\pi A/(2\pi)^2\hbar](-i)^\mu \left(\frac{l'!}{(l'+|m'|)!} \frac{l!}{(l+|m|)!} \right)^{1/2}$$

$$\times \int_0^\infty dx \, \exp(-x^2) x^{m'+m+1} L_{l'}^{m'}(x^2) L_l^m(x^2)$$

$$\times [\Theta(a - bx) K_\mu(a(\alpha^2 + q^2)^{1/2}) I_\mu(b(\alpha^2 + q^2)^{1/2}x)$$

$$+ \Theta(bx - a) I_\mu(a(\alpha^2 + q^2)^{1/2}) K_\mu(b(\alpha^2 + q^2)^{1/2}x)] \qquad (17)$$

where Θ denotes the Heaviside step function while I_μ and K_μ represent modified Bessel and Hankel functions respectively. In our present paper we shall not pursue this way of evaluation of the matrix elements $M_{fi}^{(n)}$ any further. We have only written down equation (17) in order to correct an error which crept in during our first investigation of the electron scattering problem (Bergou *et al* 1982, formula 3.23.a).

4. Considerations of the classical limit

In order to analyse the exact results for Compton scattering and for electron scattering as expressed by equations (14), (14a) and (16) respectively, we shall first make some remarks on the 'classical meaning' of the quantum numbers which characterise the transverse motion of an electron in a homogeneous magnetic field. Since classically an electron can never have a negative component of angular momentum in the magnetic-field configuration considered, we shall take $m > 0$. For large positive values, the principal quantum number $l + m$ determines the classical radius of gyration ρ_c via the relation (Johnson and Lippman 1949)

$$b^2(l + m) = \rho_c^2 \qquad b^2 = \gamma^{-1} = 2\hbar/M\omega_c. \qquad (18a)$$

On the other hand, the radial position of the centre of electron gyration (the guiding centre) ρ_0 is determined by the radial quantum number l and the corresponding relation reads

$$b^2 l = \rho_0^2. \qquad (18b)$$

This classical interpretation of the quantum numbers $l + m$ and l can also be justified

by investigating the properties of coherent superpositions of Landau states of the transverse electron motion (Varró 1984). From equations (18a, b) we conclude that $\rho_0 = \rho_c[l/(l+m)]^{1/2}$. If, therefore, we consider the classical limit which corresponds to highly excited quantum states ($l+m \to \infty$) then for the energy of these states we shall have $\hbar\omega_c(l+m) = Mv_\perp^2/2$ and this energy has to be kept in the limit at a fixed value. Consequently, ρ_c also has a fixed value and we obtain for ρ_0 the condition

$$\rho_0 = \rho_c\varepsilon \qquad \varepsilon \equiv \lim_{\substack{l+m \to \infty \\ l \to \infty}} (l/l+m)^{1/2} < 1. \tag{19}$$

In this equation the value of ε depends on how we take the limit.

As has been shown in our foregoing paper (Varró and Ehlotzky 1984) the classical differential cross section formulae for modified Thomson scattering in the external background fields equation (1), which we have derived earlier (Varró *et al* 1984), can be recovered from equations (14) and (14a) if from the very beginning we take $l' = l = 0$. This last condition corresponds to completely neglecting the transverse quantum recoil during the scattering process. In general this recoil will lead to a change of the radial position of the guiding centre.

For a classical electron the z-component of the canonical angular momentum $L_z \equiv (x \times p)_z$ and of the kinetic angular momentum $L_z^{kin} \equiv M(x \times v)_z$ can only be constants of motion simultaneously if the centre of gyration coincides with the origin of the reference frame. In the quantum mechanical problem, on the other hand, the states $|lm\rangle$ are eigenstates of the canonical angular momentum operator $\hat{L}_z$ and at the same time these states correspond to a certain radial position of the centre of gyration which is determined by the radial quantum number l. This shows that the classical correspondence can be maintained by either considering states with $l = 0$ or by putting $\varepsilon = 0$ in the quantum mechanical cross section formulae after the limit $l \to \infty$ has been taken, keeping $\lambda = l' - l$ to a fixed value.

In the following we shall therefore approach the classical limit in our two scattering problems by considering the Hilb-type asymptotic formula of the associated Laguerre polynomials (Erdély 1953). According to the Hilb formula the functions $I_{l,\lambda}$ which are defined by equation (14a) can be written as

$$I_{l,\lambda} = J_\lambda(2\beta l^{1/2}) + O(l^{-3/4}) \tag{20a}$$

and

$$I_{l+m,\nu} = J_\nu(2\beta(l+m)^{1/2}) + O((l+m)^{-3/4}). \tag{20b}$$

4.1. Classical limit of Compton scattering

If we introduce in equation (14) the relations equations (18a, b) and the asymptotic formulae equations (20a, b) and if we then let $l+m \to \infty$ and $l \to \infty$ while keeping ν and λ to fixed values at the same time, then the differential cross sections for modified Compton scattering can be brought to the form

$$d\sigma_{n\nu} = d\sigma_{Th}(\Omega_{n\nu}/\Omega)J_n^2(Q_\perp a)J_\nu^2(Q_\perp\rho_c)J_\lambda^2(Q_\perp\rho_0) \tag{21}$$

where all the parameters $Q_\perp$, a, ρ_c and ρ_0 which appear on the right-hand side of this equation have a well defined classical meaning. However, the last factor, $J_\lambda^2(Q_\perp\rho_0)$, in equation (21) cannot be of classical origin. It accounts for the quantum recoil effects. This is so, even though the argument of this Bessel function contains ρ_0 as a classical macroscopic quantity, however, the change of its value is determined by the

finite index λ which expresses the fact that during the scattering process ρ_0 varies on the quantum scale. It may be surprising that the cross sections of modified Compton scattering depend on the positions of the centre of gyration of the scatterer. However, this can easily be understood if we remember that the quantum number l determines the radial distance of the guiding centre from the origin but it does not specify its azimuthal position on the circumference of the circle of radius ρ_0. In other words, the state $|lm\rangle$ describes a smeared out charge distribution of disc-like shape for which the total radius is $\rho_0 + \rho_c$. Consequently it appears natural that ρ_0 shows up in the recoil term. If we sum over λ, which means summing over all recoil factors we get back to the classical formula (Varró *et al* 1984, Varró and Ehlotzky 1984)

$$d\sigma_n^{cl} = d\sigma_{Th}(\Omega_{n\nu}/\Omega)J_n^2(Q_\perp a)J_\nu^2(Q_\perp \rho_c). \tag{22}$$

On the other hand, if we keep in mind that according to equation (19) we have $\rho_0 = \rho_c \varepsilon$, then, by putting $\varepsilon = 0$, we obtain for the last Bessel function in equation (21), $J_\lambda(0) = \delta_{\lambda,0}$, and we therefore recover once more the classical result.

4.2. Classical limit of electron scattering

Let us now turn to the problem of electron scattering. Here we shall again use the Hilb-type formulae equations (20a, b) and we shall take into account the definitions equations (18a, b). Then the matrix elements $M_{fi}^{(n)}$ of equation (16) can be written in the form

$$M_{fi}^{(n)} = \delta_{n,\mu}(-1)^\mu (A/\pi\hbar) \int_0^\infty k_\perp \, dk_\perp \frac{J_\mu(k_\perp a)}{\alpha^2 + q^2 + k_\perp^2} J_{n+\lambda}(k_\perp \rho_c)J_\lambda(k_\perp \rho_0). \tag{23}$$

Using the formula (6.541.1) of Gradshteyn and Ryzhik (1980) we can explicitly perform the integration on the right-hand side of equation (23) to obtain the following closed-form expression for the matrix elements

$$M_{fi}^{(n)} = \delta_{n\mu}(-1)^n (A/\hbar\pi)$$
$$\times \begin{cases} (-1)^\lambda K_n(|a|(\alpha^2+q^2)^{1/2})I_{n+\lambda}(\rho_c(\alpha^2+q^2)^{1/2})I_\lambda(\rho_0(\alpha^2+q^2)^{1/2}) \\ I_n(|a|(\alpha^2+q^2)^{1/2})K_{n+\lambda}(\rho_c(\alpha^2+q^2)^{1/2})I_\lambda(\rho_0(\alpha^2+q^2)^{1/2}) \end{cases} \tag{24}$$

where the upper part of this formula holds for $|a| > \rho_c + \rho_0$ and the lower part for $|a| < \rho_c - \rho_0$. Moreover, I_n and K_n denote respectively modified Bessel and Hankel functions of the order n.

From the matrix elements, equation (24), we obtain for the relative number of reactions during the scattering process (i.e. the number of scattered particles over the number of incoming particles) R the formula

$$R_{n,\lambda} = [(2\pi)^2/v_z'v_z]|M_{fi}^{(n)}|^2 = (4A^2/\hbar^2 v_z'v_z)$$
$$\times \begin{cases} K_n^2(|a|(\alpha^2+q^2)^{1/2})I_{n+\lambda}^2(\rho_c(\alpha^2+q^2)^{1/2})I_\lambda^2(\rho_0(\alpha^2+q^2)^{1/2}) & |a| > \rho_c + \rho_0 \\ I_n^2(|a|(\alpha^2+q^2)^{1/2})K_{n+\lambda}^2(\rho_c(\alpha^2+q^2)^{1/2})I_\lambda^2(\rho_0(\alpha^2+q^2)^{1/2}) & |a| < \rho_c - \rho_0. \end{cases} \tag{25}$$

Here we have introduced $v_z = |p_z|/M$ and $v_z' = |p_z'|/M$. The frequency of transitions $w_{n,\lambda}$ as the number of scattering processes per unit time is then given by

$$w_{n,\lambda} = C_{zi}R_{n,\lambda} \tag{26}$$

where C_{zi} is the flux of ingoing particles. If C_{zi} is equal to unity then $R_{n,\lambda}$ is just the

transition probability per unit time (transition rate). We should also point out that in our final results equations (24) and (25) the parameter $|a|$ appears which is the modulus of the parameter a defined by equation (6). Moreover, we have to keep in mind that on account of the energy conservation relation, expressed by the δ function in equation (10), the momentum transfer $\hbar q$ will depend on n and λ.

Finally we consider electron scattering in the classical 'transverse recoil-free' limit. Then the transition probabilities equation (25) reduce to simpler expressions on account of the relation $I_\lambda(0) = \delta_{\lambda,0}$ for $\rho_0 = \rho_c \varepsilon$ with $\varepsilon = 0$. In order to discuss this special case somewhat further we shall specify the constant A in the scattering potential equation (3) by putting $A = Ze^2$ and we shall assume that $|\Delta\omega| < \omega, \omega_c$.

The validity of the Born approximation which has been used in equation (4) rests on the condition $Ze^2/\hbar c \ll \beta_z$ where $\beta_z = v_z/c$. By means of the energy conservation relation for our scattering process we can express the momentum transfer $\hbar q_n = \hbar(n\Delta\omega/\bar{v}_z)$ where $\bar{v}_z = (v_z + v'_z)/2$. Taking into account our above conditions for β_z and $|\Delta\omega|$ we can easily show that we must require

$$|q_n| \ll n/(Ze^2/\hbar c)\lambda \ll n\alpha/(Ze^2/\hbar c) \tag{27}$$

and therefore in this case the relative number of reactions R_n can be expressed in the form

$$R_n = (Ze^2/\hbar c)^2(4/\beta^2)\begin{cases} K_n^2(|a|\alpha)I_n^2(\rho_c\alpha) & |a| > \rho_c \\ I_n^2(|a|\alpha)K_n^2(\rho_c\alpha) & |a| < \alpha_c. \end{cases} \tag{28}$$

We now remember the simple physical meaning of the parameter $|a|$ which appears in the arguments of the Bessel and Hankel functions of equation (28) and which has been defined in equation (6). This parameter was found to be the amplitude of that component of the electron's transverse motion in the external field equation (1) which corresponds to the oscillation with the microwave frequency ω. The total transverse motion of the electron is then given by the superposition of this quivering motion and the electron's cyclotron motion along a circle of radius ρ_c.

Finally, we can get an even simpler expression for R_n if we assume that the characteristic lengths $|a|$ and ρ_c are much larger than the screening length α^{-1} of the scattering potential, equation (3). This condition can be satisfied within a wide range of parameter values of the external field, equation (1). In this case we can deduce from equation (28) the simplified formula

$$R_n = (Ze^2/\hbar c)^2(2/\beta^2)(|a|\alpha\rho_c\alpha)^{-1/2}\exp(-|\rho_c - |a||\alpha) \tag{29}$$

with $|a|\alpha, \rho_c\alpha \gg 1, n$. This result follows from the asymptotic expansions for I_n and K_n for large values of the arguments. $|\rho_c - |a||$ represents the radial distance of the nearest points of encounter of the electron's trajectory with the scattering centre. Therefore it is intuitively clear that for values $|\rho_c - |a||$ much larger than the screening length α^{-1} the effect of the scatterer can be neglected. This fact is also born out by equation (29) since according to this formula the number of reactions becomes exponentially small.

5. Conclusions

In the foregoing sections we have made a parallel investigation of the scattering of light by a free electron (non-relativistic Compton scattering), represented by the

diagram of figure 1, and of electron scattering by a static screened Coulomb potential, shown in figure 2, in the simultaneous presence of an external field configuration equation (1) composed of a circularly polarised microwave and a homogeneous magnetic field. In the classical limit, in particular, which is extensively discussed in § 4, these two scattering processes and the non-linear effects induced by the external fields are very closely related to each other and for both cases closed analytic expression can be derived for the scattering probabilities and cross sections which are given by equations (22) and (28) respectively. These results confirm our conjectures which we have made in our previous work (Varró *et al* 1984) according to which the solution of the light scattering problem paves the way for the corresponding treatment of electron scattering. This is particularly true for the consideration of the classical limit. Our more general formulae, equations (21) and (25), can be discussed for a wide range of parameter values for α, a, ρ_c, ρ_0, n, ν and λ which will be done in more detail in a forthcoming publication.

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