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Gauge-invariant relativistic Wigner functions

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Abstract
On the basis of the Hamiltonian form of the Klein–Gordon equation for a charged scalar particle field introduced by Feshbach and Villars, the gauge-invariant 2×2 Wigner matrix has been constructed whose diagonal elements describe positive and negative charge densities and whose off-diagonal elements correspond to cross-densities in phase space. The system of coupled transport equations has been derived in the case of interaction with an arbitrary external electromagnetic field. A gauge-independent generalization of the free particle representation due to Feshbach and Villars is given, and on the basis of it both the nonrelativistic and the classical limits of the relativistic quantum Boltzmann–Vlasov equation are discussed. In the nonrelativistic limit ($p/mc \to 0$) the set of equations of motion decouple to two independent quantum transport equations describing the dynamics of oppositely charged positon and negaton densities. In the classical limit ($\hbar \to 0$) two relativistic Boltzmann–Vlasov equations result for the diagonal positon and negaton densities. Even though the Planck constant $\hbar$ is absent from the latter equations, the real part of the positon–negaton cross-density does not vanish.

Keywords: Gauge invariance, relativistic scalar particles, Wigner functions

1. Introduction

For some time now, the gauge-invariant relativistic Wigner functions have been the subject of a growing research activity [1–6]. The quantum corrections to the classical nonrelativistic and relativistic transport equations can be conveniently studied by using the equations of motion in phase space for the corresponding Wigner functions. In this way the classical intuition may be taken over to quantum mechanics even in the relativistic domain.

It is known [3] that for a nonrelativistic charged particle the gauge-invariant Wigner function has to be defined as

$$W(\boldsymbol{r}, \boldsymbol{k}; t) \equiv (2\pi\hbar)^{-3} \int \mathrm{d}^3u\, \psi^*\left(\boldsymbol{r} + \tfrac{1}{2}\boldsymbol{u}, t\right)\psi\left(\boldsymbol{r} - \tfrac{1}{2}\boldsymbol{u}, t\right) \times \exp\left\{\frac{\mathrm{i}}{\hbar}\boldsymbol{u}\cdot\left[\boldsymbol{k} + \frac{e}{c}\int_{-1/2}^{+1/2} \mathrm{d}s\, \boldsymbol{A}(\boldsymbol{r} + s\boldsymbol{u}, t)\right]\right\} \quad (1)$$

where $\boldsymbol{k}$ here denotes the kinetic momentum variable. Thanks to the line integral in the exponent, this function preserves its form under the simultaneous gauge transformations of the first kind $\psi \to \psi' = \exp\left(\frac{\mathrm{i}e}{\hbar c}\chi\right)\psi$ and of the second kind $\boldsymbol{A} \to \boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\chi$, where χ is the generating function of these transformations. The manifestly covariant relativistic gauge-invariant Wigner function of a charged scalar particle was constructed long ago by Stratonovich [7]:

$$W(x, p) \equiv (2\pi\hbar)^{-4} \int \mathrm{d}^4y\, \Phi^*\left(x + \tfrac{1}{2}y, t\right)\Phi\left(x - \tfrac{1}{2}y, t\right) \times \exp\left\{-\frac{\mathrm{i}}{\hbar}y\left[p + \frac{e}{c}\int_{-1/2}^{+1/2} \mathrm{d}s\, A(x + sy, t)\right]\right\}. \quad (2)$$

For the Dirac particles studied by Vasak *et al* [2] a similar construction leads to a 4×4 Wigner matrix:

$$W(x, p) \equiv (2\pi\hbar)^{-4} \int \mathrm{d}^4y\, \Psi\left(x - \tfrac{1}{2}y, t\right)\bar{\Psi}\left(x + \tfrac{1}{2}y, t\right) \times \exp\left\{-\frac{\mathrm{i}}{\hbar}y\left[p + \frac{e}{c}\int_{-1/2}^{+1/2} \mathrm{d}s\, A(x + sy, t)\right]\right\} \quad (3)$$

whose components can be expressed by 16 real phase-space densities W_0, W_μ, W_5, $W_{\mu5}$, and $W_{\mu\nu}$ where μ and ν are spinor indices. The common feature of these latter two expressions is that the integration with respect to y goes over the whole of space–time. Accordingly, the Wigner functions depend on the four positions x and on the four kinetic momenta p. As has been shown [2, 7], in deriving the covariant equations of motion one encounters in addition certain constraint equations which are generalizations of the mass–shell relation in phase space. In the following, we use the Hamiltonian description rather than the covariant formulation.

The motivation of this work is a desire to develop the relativistic phase-space description of the dynamics of charged Klein–Gordon (KG) particles analogously to the Hamiltonian description used earlier by Bialynicki-Birula *et al* [1] for the Dirac particles. The advantage of this method compared with the manifestly covariant descriptions [2, 7] is that a single time parameter is used to describe the dynamics. In section 2, we give the gauge-invariant definition of the 2×2 Wigner matrix of charged scalar particles, on the basis of the Feshbach–Villars Hamiltonian formulation of the KG equation [8], and summarize its most important physical properties. In section 3 we present the gauge-independent relativistic quantum Boltzmann–Vlasov equation (RQBVE) for the Wigner matrix and derive the coupled set of equations of motion for the four real phase-space distributions related to diagonal and cross-densities of KG particles and their antiparticles (positons and negatons). In section 4 we study the nonrelativistic limit ($p/mc \to 0$) and the relativistic classical limit ($\hbar \to 0$) of the RQBVE. In the latter case we derive an explicit form for the Wigner matrix in terms of the classical distribution functions of positons and negatons.

2. Feshbach–Villars formulation of the Klein–Gordon equation and the gauge-invariant Wigner matrix of charge scalar particles

The KG equation of a charged scalar field of mass m and of charge e,

$$\left[\left(\mathrm{i}\hbar\,\partial - \frac{e}{c}A\right)^2 - (mc)^2\right]\Phi = 0, \tag{4}$$

is a second-order equation with respect to the time derivative. We use the convention for the metric $g = \mathrm{diag}(+,-,-,-)$ and the notation $\partial = \{\partial_\mu\} = \partial/\partial x^\mu$, $\{x^\mu\} = (ct, \boldsymbol{r})$, and $A = \{A^\mu\} = (A_0, \boldsymbol{A})$. The four product is denoted by $a \cdot b = a_\mu \cdot b^\mu$ and $a^2 = a \cdot a$. In order to get the Hamiltonian form of equation (4) we follow Feshbach and Villars [8] and introduce the two-component wavefunction

$$\Psi \equiv \begin{pmatrix}\phi\\ \chi\end{pmatrix}, \qquad \phi, \chi \equiv \frac{1}{\sqrt{2}}\left(\Phi \pm \frac{1}{mc}\Pi_0\Phi\right), \qquad \Pi_0 \equiv \mathrm{i}\hbar\,\partial_0 - \frac{e}{c}A_0, \tag{5}$$

in terms of which (4) can be written as

$$\mathrm{i}\hbar\frac{\partial\Psi}{\partial t} = \left[(\tau_3 + \mathrm{i}\tau_2)\frac{1}{2m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2 + eA_0 + \tau_3 mc^2\right]\Psi. \tag{6}$$

In equation (6) and henceforth the notation

$$\tau_0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad \tau_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \tau_2 = \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}, \qquad \tau_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{7}$$

is used for the Pauli matrices. In the framework of the Hamiltonian description, the charge density (in units of e) and the normalization conditions can be expressed as

$$\rho = \phi^*\phi - \chi^*\chi = \Psi^+\tau_3\Psi \equiv \bar{\Psi}\cdot\Psi, \qquad \int \mathrm{d}^3r\,\bar{\Psi}\cdot\Psi = \pm 1, \tag{8}$$

where $\bar{\Psi} = \Psi^+\tau_3$ denotes the Feshbach–Villars adjoint of the wavefunction Ψ. We see that the charge density is the difference of two positive definite densities. The $\pm$ signs on the right-hand side of the normalization condition correspond to positive and negative energy solutions; in other words, they refer to quantum states of positons and negatons, respectively.

Now we define the gauge-invariant Wigner matrix in the Feshbach–Villars representation:

$$W(\boldsymbol{r}, \boldsymbol{p}; t) \equiv (2\pi\hbar)^{-3}\int \mathrm{d}^3y\,\Psi\big(\boldsymbol{r} - \tfrac{1}{2}\boldsymbol{y}, t\big)\bar{\Psi}\big(\boldsymbol{r} + \tfrac{1}{2}\boldsymbol{y}, t\big) \times \exp\left\{\frac{\mathrm{i}}{\hbar}\boldsymbol{y}\cdot\left[\boldsymbol{p} + \frac{e}{c}\int_{-1/2}^{+1/2}\mathrm{d}s\,\boldsymbol{A}(\boldsymbol{r} + s\boldsymbol{y}, t)\right]\right\}. \tag{9}$$

The vector parameter $\boldsymbol{p}$ denotes the (gauge-invariant) kinetic momentum, and the line integral on the right-hand side of (9) secures the gauge invariance of W, as has been shown e.g. in [3]. We remark that the original scalar wavefunction Φ is invariant under the *pure Lorentz transformation* $\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{\epsilon}ct$, $ct' = ct - \boldsymbol{\epsilon}\cdot\boldsymbol{r}$, i.e. $\Phi'(\boldsymbol{r}', t') = \Phi(\boldsymbol{r}, t)$, where $\boldsymbol{\epsilon}$ is an infinitesimal vector. On the other hand, owing to the definition (5) and the transformation rule $\Pi_0' = \Pi_0 - \boldsymbol{\epsilon}\cdot\boldsymbol{\Pi}$, the two-component wavefunction Ψ transforms like $\Psi'(\boldsymbol{r}', t') = [1 - (\tau_3 + \mathrm{i}\tau_2)\boldsymbol{\epsilon}\cdot\boldsymbol{\Pi}/2mc]\Psi(\boldsymbol{r}, t)$, as given in the book by deGroot and Suttorp [9], where $\boldsymbol{\Pi} = -\mathrm{i}\hbar\,\partial/\partial\boldsymbol{r} - (e/c)\boldsymbol{A}$ denotes the kinetic momentum operator. Henceforth, the Wigner matrix (9) cannot be form invariant under the Lorentz transformation. We leave the detailed discussion of the transformation properties of the Wigner matrix and of its equation of motion to a separate publication. By construction, W is a self-adjoint matrix in the Feshbach–Villars sense, namely $\bar{W} \equiv \tau_3 W^+\tau_3 = W$, and its explicit form can be expressed in terms of the components ϕ and χ, yielding

$$2W = 2\begin{pmatrix}W_{\phi\phi} & -W^*_{\phi\chi}\\ W_{\phi\chi} & -W_{\chi\chi}\end{pmatrix} = \tau_0 W_0 + \mathrm{i}\tau_1 W_1 - \mathrm{i}\tau_2 W_2 + \tau_3 W_3. \tag{10}$$

In equation (10), W_0, W_1, W_2, W_3 are real and can be expressed as

$$\begin{pmatrix}W_0\\ W_1\\ W_2\\ W_3\end{pmatrix} = \mathrm{Tr}\left[W\begin{pmatrix}\tau_0\\ -\mathrm{i}\tau_1\\ \mathrm{i}\tau_2\\ \tau_3\end{pmatrix}\right]. \tag{11}$$

It can be easily checked that $W_0 = W_{\phi\phi} - W_{\chi\chi}$, $W_1 = 2\,\mathrm{Im}[W_{\phi\chi}]$, $W_2 = 2\,\mathrm{Re}[W_{\phi\chi}]$ and $W_3 = W_{\phi\phi} + W_{\chi\chi}$. W_0 and W_3 are proportional to the charge density and mass density in phase space, respectively, showing that the centre of charge and that of mass do not necessarily coincide. The sum of W_2 and W_3 is related to the current density:

$$j(r,t)=\int \mathrm{d}^3p\, p[W_2(r,p,t)+W_3(r,p,t)]. \tag{12}$$

The imaginary part W_1 of the cross-density $W_{\phi\chi}$ can be directly related to the 'Zitterbewegung'. This can be seen in the simplest way by a direct calculation of $I(p,t)=\int \mathrm{d}r^3\, W(r,p,t)$ for an arbitrary normalizable free particle wavepacket. For pure positon or pure negaton states, $I(p,t)$ does not depend on time and it is real; hence $I_1=0$. On the other hand, for superpositions of positon and negaton states, $I_1\neq 0$; moreover, $I_1(p,t)$ oscillates with the frequency $2\omega_c\sqrt{1+(p/mc)^2}$, where $\omega_c=mc^2/\hbar$ is the Compton frequency. This is just the 'Zitterbewegung' or 'trembling motion'. As simple illustrations, we give the Wigner matrices of a free positon and a free negaton plane wave of momentum q:

$$W_q^+=\frac{1}{V}\delta_3(p-q)\begin{pmatrix}\cosh^2\xi & \frac{1}{2}\sinh 2\xi\\ -\frac{1}{2}\sinh 2\xi & -\sinh^2\xi\end{pmatrix}, \tag{13}$$

$$W_q^-=\frac{1}{V}\delta_3(p-q)\begin{pmatrix}\sinh^2\xi & \frac{1}{2}\sinh 2\xi\\ -\frac{1}{2}\sinh 2\xi & -\cosh^2\xi\end{pmatrix}, \tag{14}$$

where $\xi=\frac{1}{2}\log\sqrt{1+(q/mc)^2}$ and V is the normalization volume.

3. The gauge-independent relativistic quantum Boltzmann–Vlasov equation of charge scalar particles

The equation of motion for the Wigner matrix (9) defined in the previous section can be derived from the Schrödinger-type equation (6) for $\Psi(r\pm\frac{1}{2}y,t)$ by using the Gauss theorem (partial integrations with vanishing surface terms) in calculating the y-integral. The calculation consists of steps completely analogous to those used earlier by us in the case of a Schrödinger particle [3]. For example, since the defining expression for W is in fact a Fourier transform, the multiplicative terms proportional to y go over to the gradient $-\mathrm{i}\hbar\,\partial/\partial p$ which can be taken out from the integral. Analogously, the gradient $-\mathrm{i}\hbar\,\partial/\partial r$ in the Hamiltonian is replaced by the parameter p. For details, we refer the reader to our earlier publication [3]. We obtain

$$(\partial_t+\hat{D}_E)W+(\hat{D}_c+\hat{D}_B)\frac{1}{2}\{\tau_3+\mathrm{i}\tau_2,W\}+\frac{\mathrm{i}}{\hbar}[H_0(\hat{P}),W]=0, \tag{15}$$

where

$$\hat{D}_E\equiv e\hat{E}\cdot\nabla_p,\qquad \hat{E}=j_0\left(\frac{\hbar}{2}\nabla_r\cdot\nabla_p\right)E(r,t), \tag{16}$$

$$\hat{D}_c\equiv\frac{1}{m}(p+\Delta\hat{p})\cdot\nabla_r,\qquad \Delta\hat{p}\equiv-\frac{e\hbar}{4c}\nabla_p\times\tilde{B}, \tag{17}$$

$$\tilde{B}=j_1\left(\frac{\hbar}{2}\nabla_r\cdot\nabla_p\right)B(r,t), \tag{18}$$

$$\hat{D}_B\equiv\frac{e}{mc}[(p+\Delta\hat{p})\times\hat{B}]\cdot\nabla_p,\qquad \hat{B}=j_0\left(\frac{\hbar}{2}\nabla_r\cdot\nabla_p\right)B(r,t), \tag{19}$$

$$\hat{P}^2\equiv(p+\Delta\hat{p})^2-\frac{\hbar^2}{4}\left(\nabla_r-\frac{e}{c}\nabla_p\times\hat{B}\right)^2, \tag{20}$$

$$H_0(\hat{P})\equiv(\tau_3+\mathrm{i}\tau_2)\frac{\hat{P}^2}{2m}+\tau_3mc^2. \tag{21}$$

In the equations above the functions $j_0(x)=\sin x/x=1-x^2/3!+x^4/5!\mp\cdots$ and $j_1(x)=-\mathrm{d}j_0(x)/\mathrm{d}x=x/3-x^3/(3!\cdot 5)+x^5/(5!\cdot 7)\mp\cdots$ are ordinary spherical Bessel functions of the first kind of order zero and one, respectively. In the argument of the Bessel functions, the gradient ∇_r acts on the electric field strength E and on the magnetic induction B, and the gradient ∇_p acts on the Wigner functions. Notice that $\hat{E}$, $\hat{B}$, and the momentum correction operator $\Delta\hat{p}$ can be expanded into a power series containing only even powers of $\hbar$. The matrix $H_0(\hat{P})$ in the limit of $\hbar\to 0$ becomes the free particle Hamiltonian $H_0(p)$ in the momentum representation. By using the decomposition $W=\mathrm{Re}\,W+\mathrm{i}\,\mathrm{Im}\,W$ in equation (15), we can derive a set of two real equations:

$$(\partial_t+\hat{D}_E)\,2\,\mathrm{Re}\,W+(\hat{D}_c+\hat{D}_B)\tfrac{1}{2}\{\tau_3+\mathrm{i}\tau_2,2\,\mathrm{Re}\,W\}-\frac{1}{\hbar}[H_0(\hat{P}),\tau_1]W_1=0, \tag{22}$$

$$(\partial_t+\hat{D}_E)\tau_1W_1+\frac{1}{\hbar}[H_0(\hat{P}),2\,\mathrm{Re}\,W]=0. \tag{23}$$

In obtaining equation (23) we have taken into account that $2\,\mathrm{Im}\,W=\tau_1W_1$, and the anticommutator $\{\tau_3+\mathrm{i}\tau_2,\tau_1\}$ vanishes.

The real part of $2W$ can be expressed as $2\,\mathrm{Re}\,W=\tau_0W_0-\mathrm{i}\tau_2W_2+\tau_3W_3$. With the help of this decomposition the following set of coupled equations can be derived for the real phase-space densities W_0, W_1, W_2, W_3:

$$(\partial_t+\hat{D}_E)W_0+(\hat{D}_c+\hat{D}_B)(W_2+W_3)=0, \tag{24}$$

$$(\partial_t+\hat{D}_E)W_2-(\hat{D}_c+\hat{D}_B)W_0+2\omega_c\left(1+\frac{\hat{P}^2}{2m^2c^2}\right)W_1=0, \tag{25}$$

$$(\partial_t+\hat{D}_E)W_3+(\hat{D}_c+\hat{D}_B)W_0-2\omega_c\frac{\hat{P}^2}{2m^2c^2}W_1=0, \tag{26}$$

$$(\partial_t+\hat{D}_E)W_1-2\omega_c\frac{\hat{P}^2}{2m^2c^2}(W_2+W_3)-2\omega_cW_2=0, \tag{27}$$

where we have introduced the Compton frequency $\omega_c=mc^2/\hbar$. We term the set of equations (24)–(27), or their matrix equivalent (15), the RQBVE. As a boundary condition for the RQBVE in free space it is natural to assume that the functions W_μ vanish sufficiently fast as $r\to\infty$ and $p\to\infty$, such that the integrals of W_μ over the whole phase space are finite.

4. The nonrelativistic limit and the classical limit of the RQBVE

In the present section we derive both the nonrelativistic limit ($p/mc\to 0$) and the relativistic classical limit ($\hbar\to 0$) of the RQBVE with the help of the generalization of the free particle representation introduced by Feshbach and Villars [8] in their classic paper on relativistic wavefunctions. In order to do that, let us introduce the matrix operator

$$U(\hat{P})=\exp[-\tau_1\hat{\xi}(\hat{P})],\qquad \hat{\xi}(\hat{P})=\tfrac{1}{2}\log\sqrt{1+\hat{P}^2/m^2c^2}, \tag{28}$$

where $\hat{P}^2$ is given by equation (20). By performing the similarity transformation generated by U on the equation of motion (15), we have

$$(\partial_t' + \hat{D}_E')W' + (\hat{D}_c' + \hat{D}_B')\frac{mc^2}{E_{\hat{P}}}\frac{1}{2}\{\tau_3 + \mathrm{i}\tau_2, W'\} + \frac{\mathrm{i}}{\hbar}E_{\hat{P}}[\tau_3, W'] = 0, \quad (29)$$

where the transformed quantities are $W' = U^{-1}WU$, $\hat{D}_E' = U^{-1}\hat{D}_E U$, etc, and

$$E_{\hat{P}} = \sqrt{m^2c^4 + c^2\hat{P}^2}. \quad (30)$$

Now let us assume that the W_μ are confined to regions in phase space in which $p/mc \ll 1$ and that the characteristic frequencies and wavenumbers are much smaller than the Compton frequency and the Compton wavelength, respectively. The latter two conditions can be symbolically written as

$$\partial_t \ll \omega_c = mc^2/\hbar, \qquad \nabla_r \ll \kappa_c = mc/\hbar. \quad (31)$$

Moreover we assume in addition that

$$Bmc\nabla_p \ll B_{cr}, \qquad Emc\nabla_p \ll E_{cr}, \quad (32)$$

where $B_{cr} = E_{cr} = m^2c^3/e\hbar$ are the critical field strengths of quantum electrodynamics. Under these conditions, $E(\hat{P}) \to mc^2, \xi(\hat{P}) \to 0$; hence $U(\hat{P}) \to 1$, and $W' \to W^{NR}$ becomes diagonal. The remaining equations can be combined to yield

$$\left\{\partial_t + (\boldsymbol{v}+\Delta\hat{\boldsymbol{v}})\cdot\nabla_r + e\left[\hat{\boldsymbol{E}} + \frac{1}{c}(\boldsymbol{v}+\Delta\hat{\boldsymbol{v}})\times\hat{\boldsymbol{B}}\right]\cdot\nabla_p\right\}F = 0, \quad (33)$$

$$\left\{\partial_t + (\boldsymbol{v}+\Delta\hat{\boldsymbol{v}})\cdot\nabla_r - e\left[\hat{\boldsymbol{E}} + \frac{1}{c}(\boldsymbol{v}+\Delta\hat{\boldsymbol{v}})\times\hat{\boldsymbol{B}}\right]\cdot\nabla_p\right\}G = 0, \quad (34)$$

where $F(\boldsymbol{r}, \boldsymbol{p}, t) = W_{\phi\phi}^{NR}(\boldsymbol{r}, \boldsymbol{p}, t)$, $G(\boldsymbol{r}, \boldsymbol{p}, t) = W_{\chi\chi}^{NR}(\boldsymbol{r}, -\boldsymbol{p}, t)$, $\boldsymbol{v} = \boldsymbol{p}/m$, and $\Delta\hat{\boldsymbol{v}} = \Delta\hat{\boldsymbol{p}}/m$. Equation (33) coincides with the nonrelativistic quantum Boltzmann–Vlasov equation derived earlier [3] by the present authors. Equation (34) refers to an oppositely charged Schrödinger particle, demonstrating that the RQBVE describes simultaneously the dynamics of both positons and negatons.

In order to consider the classical limit of the relativistic dynamics in phase space we first observe that

$$\hat{D}_E \to e\boldsymbol{E}\cdot\nabla_p, \qquad \hat{D}_B \to \frac{e}{mc}(\boldsymbol{p}\times\boldsymbol{B})\nabla_p, \qquad U(\hat{P}) \to U(p) \quad (35)$$

$$\hat{D}_c \to \frac{1}{m}\boldsymbol{p}\cdot\nabla_r, \qquad \hat{P}^2 \to p^2, \qquad H_0(\hat{P}) \to H_0(p) \quad (36)$$

in the $\hbar \to 0$ limit. Equation (22) goes over to a meaningful limit equation only if $W_1 \to 0$ faster than $\hbar \to 0$. Hence $W^{cl} = \mathrm{Re}\, W^{cl}$. We note that a more rigorous treatment of the $\hbar \to 0$ limit can be carried out by the coarse-graining technique introduced by Shin and Rafelski [5] in the case of Dirac electrons. On performing the similarity transformation $W' = U^{-1}(p)W^{cl}U(p)$ generated by the matrix $U(p)$, we have

$$(\partial_t + e\boldsymbol{E}\cdot\nabla_p)W' - e\boldsymbol{E}\cdot\frac{\boldsymbol{p}c^2}{2E_p^2}[\tau_1, W'] + \frac{mc^2}{E_p}\left[\frac{1}{m}\boldsymbol{p}\cdot\nabla_r + \frac{e}{mc}(\boldsymbol{p}\times\boldsymbol{B})\cdot\nabla_p\right] \times \frac{1}{2}\{\tau_3 + \mathrm{i}\tau_2, W'\} = 0, \quad (37)$$

$$[\tau_3, W'] = 0. \quad (38)$$

According to equation (38), W' must be diagonal; hence, using the notation $W'_{\phi\phi}(\boldsymbol{r}, \boldsymbol{p}, t) = f$ and $W'_{\chi\chi}(\boldsymbol{r}, -\boldsymbol{p}, t) = g$, we have

$$W^{cl} = U(p)\begin{pmatrix} f & 0 \\ 0 & -g \end{pmatrix}U^{-1}(p) = f\begin{pmatrix} \cosh^2\xi & \frac{1}{2}\sinh 2\xi \\ -\frac{1}{2}\sinh 2\xi & -\sinh^2\xi \end{pmatrix} + g\begin{pmatrix} \sinh^2\xi & \frac{1}{2}\sinh 2\xi \\ -\frac{1}{2}\sinh 2\xi & -\cosh^2\xi \end{pmatrix} \quad (39)$$

with $\xi = \frac{1}{2}\log\sqrt{1 + (p/mc)^2}$. From the off-diagonal part of equation (37), the following balance equation can be derived:

$$\int_V \mathrm{d}^3r \int \mathrm{d}^3p\, e\boldsymbol{E}\cdot\boldsymbol{v}(\boldsymbol{p})(f - g) = -\int_S \mathrm{d}\boldsymbol{s}\cdot\int \mathrm{d}^3p\, \gamma mc^2\boldsymbol{v}(\boldsymbol{p})(f + g). \quad (40)$$

This means that the work done by the electric field on the positon–negaton field per unit time in a volume V equals the inward flux of the energy current density over the bounding surface S. The equation of motion for the classical phase-space distribution can be derived from equation (37):

$$\partial_t f + \boldsymbol{v}(\boldsymbol{p})\cdot\nabla_r f + e\left[\boldsymbol{E} + \frac{1}{c}(\boldsymbol{v}(\boldsymbol{p})\times\boldsymbol{B})\right]\cdot\nabla_p f = 0, \quad (41)$$

$$\partial_t g + \boldsymbol{v}(\boldsymbol{p})\cdot\nabla_r g - e\left[\boldsymbol{E} + \frac{1}{c}(\boldsymbol{v}(\boldsymbol{p})\times\boldsymbol{B})\right]\cdot\nabla_p g = 0, \quad (42)$$

where

$$\boldsymbol{v}(\boldsymbol{p}) = \frac{\boldsymbol{p}/m}{\sqrt{1 + (p/mc)^2}} = \boldsymbol{p}/m\gamma \quad (43)$$

is the velocity function, and $\gamma = 1/\sqrt{1 - v^2/c^2}$ denotes the usual relativistic factor.

As is seen from equations (41) and (42), in the classical limit the RQBVE can be reduced to two uncoupled relativistic Boltzmann–Vlasov equations describing the phase-space dynamics of positons and negatons. According to equation (39), in the case of positons we take $g = 0$ and we have

$$W_+^{cl} = f\begin{pmatrix} \cosh^2\xi & \frac{1}{2}\sinh 2\xi \\ -\frac{1}{2}\sinh 2\xi & -\sinh^2\xi \end{pmatrix}, \qquad \int \mathrm{d}^3r\,\mathrm{d}^3p\,\mathrm{Tr}[W_+^{cl}] = +1. \quad (44)$$

Similarly for a negaton solution we have

$$W_-^{cl} = g\begin{pmatrix} \sinh^2\xi & \frac{1}{2}\sinh 2\xi \\ -\frac{1}{2}\sinh 2\xi & -\cosh^2\xi \end{pmatrix}, \qquad \int \mathrm{d}^3r\,\mathrm{d}^3p\,\mathrm{Tr}[W_-^{cl}] = -1. \quad (45)$$

Finally, it is interesting to note that the real part of the cross-density W_2 does not vanish in the relativistic classical limit. The physical significance of this behaviour will be discussed elsewhere.

5. Summary

In order to consider the Hamiltonian description of charged scalar particles, we have introduced, in section 2, the Feshbach–Villars formulation of the KG equation and defined the gauge-invariant 2×2 Wigner matrix. In section 3 we derived the gauge-independent RQBVE equation for this Wigner matrix which leads to a set of four equations for four real phase-space densities whose physical meanings have been given. In section 4 we showed that the RQBVE reduces to two uncoupled quantum transport equations for the positon and the negaton in the nonrelativistic limit. In the classical limit $\hbar \to 0$, one arrives at two uncoupled relativistic Vlasov equations for positon and negaton phase-space densities; however, there remains a classical constraint equation (which does not contain $\hbar$). This means that a connection between the positon and negaton densities survives the classical limit. Finally, we have written down the general form (39) for the Wigner matrix in the classical limit.

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