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Weyl–Wigner correspondence in two space dimensions

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We consider Wigner functions in two space dimensions. In particular, we focus on Wigner functions corresponding to energy eigenstates of a non-relativistic particle moving in two dimensions in the absence of a potential. With the help of the Weyl–Wigner correspondence we first transform the eigenvalue equations for energy and angular momentum into phase space. As a result we arrive at partial differential equations in phase space which determine the corresponding Wigner function. We then solve the resulting equations using appropriate coordinates.

1. Introduction

Richard Feynman taught us that interference of probability amplitudes is a central lesson of quantum mechanics [1]. Sir Peter Knight was instrumental in developing the Wigner function [2] into a useful tool in quantum optics [3] and as an inexhaustible source of insight [4] into various manifestations of quantum interference. Apart from a few examples [5] the literature has focused on Wigner functions [6, 7] of one-dimensional quantum systems. In the present paper we derive the Wigner function of an energy eigenstate of a non-relativistic free particle in two space dimensions.

Three arguments for this restriction to two space dimensions offer themselves: (i) in two dimensions the angular momentum vector has only one component. As a result the square of the angular momentum operator is solely determined by that component. (ii) The interference of waves in two dimensions is very different from the one in three space dimensions. The violation of Huygens principle [8] in two dimensions is a manifestation of this subtle influence of the dimensions of space on wave phenomena. (iii) In the Schrödinger formulation of quantum mechanics these effects manifest themselves in an attractive centrifugal potential [9, 10] for vanishing angular momentum. In the Wigner formulation these interference forces correspond

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to non-classical correlations [11] between position and momentum reflecting themselves in negative domains of phase space.

In the energy wave functions with angular momentum $m=0$ the attraction towards the centre manifests itself as a bunching of nodes [9] towards the origin of space. For higher order angular momentum quantum numbers there is a repulsive potential and the nodes of the energy wave functions are anti-bunched at the origin. It is therefore interesting to understand how these phenomena translate into phase space.

One possibility is to start from the well-known wave function

$$\psi_m(r,\varphi) = N J_m(k_0 r)\exp(\mathrm{i}m\varphi) \tag{1}$$

of an energy eigenstate with energy $E \equiv (\hbar k_0)^2/(2M)$ in polar coordinates $r \equiv (x^2+y^2)^{1/2}$ and φ with $x = r\cos\varphi$ and $y = r\sin\varphi$. Here J_m denotes the ordinary Bessel function [12] of mth order with $N = (M/(2\pi\hbar^2))^{1/2}$ and M is the mass of the particle.

We can substitute this expression into the standard definition

$$W_\rho(\mathbf{r},\mathbf{p}) \equiv \frac{1}{(2\pi\hbar)^2}\int \mathrm{d}^2\xi\,\exp(-\mathrm{i}\mathbf{p}\cdot\xi/\hbar)\langle\mathbf{r}+\tfrac{1}{2}\xi|\hat{\rho}|\mathbf{r}-\tfrac{1}{2}\xi\rangle \tag{2}$$

of the Wigner function corresponding to the density operator $\hat{\rho}$ and perform the integration.

In the present paper we do not follow this approach since the integrals are not easy to perform. More importantly, we want to derive the Wigner function corresponding to the wave function directly from phase space. Indeed, the integrational approach seems to suggest that the Wigner function always needs the quantum state in the form of the wave function or the density operator, that is it relies solely on the Schrödinger formulation. However, there exist partial differential equations [13] in phase space which determine the Wigner function. There is no need to go through the wave function representation first.

There exists an extensive literature calculating Wigner functions from phase space [14]. However, almost all of these examples deal with one-dimensional systems. In the present paper we focus on the first nontrivial extension by calculating the Wigner function of an energy eigenfunction of a free particle in two dimensions.

Our paper is organized as follows: in section 2 we prepare the stage for our work and summarize the essential ingredients of the Wigner function phase space approach. Here we focus on the Weyl–Wigner correspondence [15, 16] of the product of two operators which arise naturally in any eigenvalue problem

$$\hat{\mathcal{L}}\hat{\rho} = \lambda\hat{\rho} \tag{3}$$

for a Hermitian operator $\hat{\mathcal{L}}$ and eigenvalue λ. We then apply this formula to the eigenvalue equations of energy of a free particle and of angular momentum in two dimensions. These equations constitute partial differential equations in phase space which are either first or second order. We dedicate section 2 to express these equations in Cartesian coordinates. Section 3 is devoted to the derivation of the allowed functional form of the Wigner function governed by the constraint equations. In section 4 we solve the eigenvalue equations for energy and angular momentum in phase space and give an explicit analytic formula for the Wigner function which still contains two integration constants. These constants are determined in section 5. A brief discussion and an outlook provide in section 6 the conclusion of the paper.

2. Phase space equations

In this section we derive the partial differential equations determining the Wigner function of an energy eigenstate $|E\rangle$ of a free particle in two dimensions. For this purpose we first establish the notation and define the eigenvalue equations for the kinetic energy and angular momentum operators. In order to simplify the equations we use as phase space variables position $\mathbf{r}$ and wave vector $\mathbf{k}$ instead of position and momentum $\mathbf{p} \equiv \hbar\mathbf{k}$. We then briefly review the concept of the Weyl–Wigner correspondence and introduce the Weyl–Wigner symbols for $\hat{\mathbf{k}}^2$ and $\hat{L}_z$. With the help of the product formula we arrive at the desired partial differential equations in phase space.

2.1 *Formulation of the problem*

In the present paper we address the problem of finding the Wigner function of an energy eigenstate $|E\rangle$ of a free particle in two dimensions. This state follows from the Schrödinger eigenvalue equation

$$\frac{\hat{\mathbf{p}}^2}{2M}|E\rangle = E|E\rangle, \tag{4}$$

where $\hat{\mathbf{p}} = \hbar/\mathrm{i}\nabla_r$ denotes the momentum operator with the gradient ∇_r in two space dimensions.

In order to simplify the equations we introduce the operator $\hat{\mathbf{k}} \equiv \hat{\mathbf{p}}/\hbar$ of the wave vector and express the energy eigenvalue

$$E \equiv (\hbar k_0)^2/(2M). \tag{5}$$

In this notation the energy eigenvalue equation reads

$$\hat{\mathbf{k}}^2|E\rangle = k_0^2|E\rangle. \tag{6}$$

The operator

$$\hat{L}_z = \hbar\left(\hat{x}\hat{k}_y - \hat{y}\hat{k}_x\right) \tag{7}$$

commutes with the kinetic energy operator. As a consequence $\hat{L}_z$ and $\hat{\mathbf{k}}^2$ have common eigenstates that is

$$\hat{L}_z|E\rangle = \hbar\left(\hat{x}\hat{k}_y - \hat{y}\hat{k}_x\right)|E\rangle = \hbar m|E\rangle, \tag{8}$$

where m is an integer. Therefore, the state $|E\rangle$ is characterized by the two quantum numbers k_0^2 and m. As a short-hand notation we denote the eigenstate by

$$|E\rangle \equiv |k_0^2, m\rangle. \tag{9}$$

When we multiply equations (6) and (8) by the bra vector $\langle E|$ we arrive at the two equations

$$\hat{\mathbf{k}}^2|E\rangle\langle E| = k_0^2|E\rangle\langle E| \tag{10}$$

and

$$\left(\hat{x}\hat{k}_y - \hat{y}\hat{k}_x\right)|E\rangle\langle E| = m|E\rangle\langle E| \tag{11}$$

consisting of the product of the density operator $\hat{\rho} \equiv |E\rangle\langle E|$ with the remnant $\hat{\mathbf{k}}^2$ of the kinetic energy operator or with the angular momentum operator.

2.2 *Weyl–Wigner correspondence*

In order to translate these equations into phase space we first recall the definition of the Weyl–Wigner correspondence

$$A(\mathbf{r}, \mathbf{k}) \equiv \int_{-\infty}^{\infty} \mathrm{d}^2\xi \, \exp(-\mathrm{i}\mathbf{k}\cdot\xi)\langle \mathbf{r} + \tfrac{1}{2}\xi|\hat{A}|\mathbf{r} - \tfrac{1}{2}\xi\rangle \tag{12}$$

of the operator $\hat{A}$. For the operators $\hat{\mathbf{k}}^2$ and $(\hat{x}\hat{k}_y - \hat{y}\hat{k}_x)$ the Weyl–Wigner symbols are identical to the form of the operators, that is they read $\mathbf{k}^2$ and $\left(xk_y - yk_x\right)$.

Moreover, we recall the product formula [7]

$$(A\cdot B)(\mathbf{r}, \mathbf{k}) = A\left(\mathbf{r} - \frac{1}{2\mathrm{i}}\frac{\partial}{\partial \mathbf{k}}, \mathbf{k} + \frac{1}{2\mathrm{i}}\frac{\partial}{\partial \mathbf{r}}\right)B(\mathbf{r}, \mathbf{k}) \tag{13}$$

for the Weyl–Wigner correspondence of the product $\hat{A}\cdot\hat{B}$ of two operators $\hat{A}$ and $\hat{B}$. Here we replace the phase space variables $\mathbf{r}$ and $\mathbf{k}$ in the Weyl–Wigner symbol by the Bopp-operators [17]

$$\mathbf{r} \to \mathbf{r} - \frac{1}{2\mathrm{i}}\frac{\partial}{\partial \mathbf{k}} \text{ and } \mathbf{k} \to \mathbf{k} + \frac{1}{2\mathrm{i}}\frac{\partial}{\partial \mathbf{r}}. \tag{14}$$

2.2.1 Energy eigenvalue equation. When we apply the product formula equation (13) to the energy eigenvalue equation, equation (6), we find

$$\left(\mathbf{k} + \frac{1}{2\mathrm{i}}\frac{\partial}{\partial \mathbf{r}}\right)^2 W_E(\mathbf{r}, \mathbf{k}) = k_0^2 W_E(\mathbf{r}, \mathbf{k}), \tag{15}$$

where

$$W_E(\mathbf{r}, \mathbf{k}) \equiv \frac{1}{(2\pi)^2}\int \mathrm{d}^2\xi \, \exp(-\mathrm{i}\mathbf{k}\cdot\xi)\langle \mathbf{r} + \tfrac{1}{2}\xi|E\rangle\langle E|\mathbf{r} - \tfrac{1}{2}\xi\rangle \tag{16}$$

denotes the Wigner function of the energy eigenstate $|E\rangle$.

We take the real and imaginary parts of equation (15) and arrive at the two equations

$$\left[\Delta_r^{(2)} + 4\left(k_0^2 - \mathbf{k}^2\right)\right] W_E(\mathbf{r}, \mathbf{k}) = 0 \tag{17}$$

and

$$\left(\mathbf{k} \cdot \frac{\partial}{\partial \mathbf{r}}\right) W_E(\mathbf{r}, \mathbf{k}) = 0, \tag{18}$$

where $\Delta_r^{(2)}$ denotes the Laplacian in the two-dimensional position space. These equations are not limited to $D=2$ but are valid in any number D of dimensions.

The first equation is the energy eigenvalue equation in phase space. Therefore, it is completely analogous to the Schrödinger eigenvalue equation. In order to distinguish equation (17) from the corresponding Schrödinger formulation we refer to it as the Weyl–Wigner energy eigenvalue equation.

The second equation is the remnant of the Liouville equation

$$\left(\frac{\partial}{\partial t} + \frac{\mathbf{p}}{M} \cdot \frac{\partial}{\partial \mathbf{r}}\right) W = 0 \tag{19}$$

for a particle in the absence of a potential which for a stationary state reduces to equation (18). This constraint on the Wigner function indicates that the gradient of W_E with respect to position is always orthogonal on the wave vector. For this reason we refer to equation (18) as the transversality condition.

2.2.2 Angular momentum eigenvalue equation. We now transform the eigenvalue equation, equation (8), for the operator $\hat{L}_z$ into phase space and find with the help of the Weyl–Wigner correspondence, equation (13), the relation

$$\left[\left(x - \frac{1}{2\mathrm{i}}\frac{\partial}{\partial k_x}\right)\left(k_y + \frac{1}{2\mathrm{i}}\frac{\partial}{\partial y}\right) - \left(y - \frac{1}{2\mathrm{i}}\frac{\partial}{\partial k_y}\right)\left(k_x + \frac{1}{2\mathrm{i}}\frac{\partial}{\partial x}\right)\right] W_E = \kappa W_E. \tag{20}$$

Here we have introduced the eigenvalue κ. An important goal is to derive the fact that κ is integer.

When we decompose this equation into real and imaginary parts we arrive at

$$\left[x k_y - y k_x + \frac{1}{4}\left(\frac{\partial^2}{\partial k_x \partial y} - \frac{\partial^2}{\partial k_y \partial x}\right)\right] W_E = \kappa W_E \tag{21}$$

and

$$\left[x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} + k_x \frac{\partial}{\partial k_y} - k_y \frac{\partial}{\partial k_x}\right] W_E = 0. \tag{22}$$

The first equation is a partial differential equation of second order and represents the Weyl–Wigner eigenvalue equation of angular momentum in phase space. The second equation is a partial differential equation of first order and represents a constraint on the Wigner function involving position as well as wave vector.

3. Solution of the constraint equations

The transversality condition, equation (18), implies that the Wigner function W_E of the energy eigenstate can only depend on x and y through the argument

$$\xi_1 \equiv xk_y - yk_x, \tag{23}$$

that is

$$W_E(\mathbf{r}, \mathbf{k}) = \mathcal{W}_t\big(xk_y - yk_x, \mathbf{k}\big), \tag{24}$$

where at the moment $\mathcal{W}_t$ could be any function that is differentiable with respect to the argument ξ_1 of $\mathcal{W}_t$ at least once. Here the subscript t reminds us that this form of the argument ξ_1 results from the transversality condition. The specific form of $\mathcal{W}_t$ will be determined by the eigenvalue equations in phase space. It is straightforward to verify that this ansatz satisfies equation (18).

Moreover, the constraint equation, equation (22), resulting from the Schrödinger eigenvalue equation of the angular momentum implies that $\mathbf{k}$ can only enter $\mathcal{W}_t$ in the form of

$$\xi_2 \equiv k_x^2 + k_y^2 = \mathbf{k}^2, \tag{25}$$

that is

$$W_E(\mathbf{r}, \mathbf{k}) = \mathcal{W}_{t-a}\big(xk_y - yk_x, \mathbf{k}^2\big). \tag{26}$$

Here the function $\mathcal{W}_{t-a}$ could be any function that is differentiable with respect to ξ_1 and with respect to the second argument ξ_2 of $\mathcal{W}_{t-a}$ at least once. Here the subscripts t and a remind us that here we have used the transversality condition as well as the constraint equation from angular momentum. The specific form of $\mathcal{W}_{t-a}$ will be determined by the Weyl–Wigner eigenvalue equations in phase space. It is straightforward to verify that this ansatz satisfies equations (18) and (22).

4. Solution of Weyl–Wigner eigenvalue equations

According to equation (26) the four phase space variables (x, y) and (k_x, k_y) enter the Wigner function W_E only through the two variables ξ_1 and ξ_2 defined by equations (23) and (25). We now use the eigenvalue equations in phase space, equations (17) and (21) to determine $\mathcal{W}_{t-a}$.

4.1 *Free particle in one dimension*

We start our analysis by first considering the Weyl–Wigner energy eigenvalue equation, equation (17). For this purpose we substitute the ansatz, equation (26), into equation (17) which yields the Schrödinger equation

$$\frac{\partial^2}{\partial \xi_1^2}\mathcal{W}_{t-a} + 4\frac{k_0^2 - \xi_2}{\xi_2}\mathcal{W}_{t-a} = 0 \tag{27}$$

of a free particle in one dimension. In our analysis of this equation we distinguish the two cases of $k_0 < |k|$ and $|k| < k_0$.

4.1.1 Forbidden domain. In the region of $k_0 < |k|$ we have to solve the differential equation

$$\left[\frac{\partial^2}{\partial \xi_1^2} - \bar{q}^2(\xi_2)\right]\mathcal{W}_{t-a}(\xi_1, \xi_2) = 0 \tag{28}$$

with

$$\bar{q}^2(\xi_2) \equiv 4\frac{\xi_2 - k_0^2}{\xi_2}. \tag{29}$$

The general solution reads

$$\mathcal{W}_{t-a}(\xi_1, \xi_2) = c_1 \exp[\bar{q}(\xi_2)\xi_1] + c_2 \exp[-\bar{q}(\xi_2)\xi_1], \tag{30}$$

where c_1 and c_2 are constants. The variable $\xi_1 = xk_y - yk_x$ can assume positive and negative values. For positive values of ξ_1 the first contribution is exponentially growing. Since the Wigner function should vanish at infinity, we find the condition $c_1 = 0$. Likewise for negative values of ξ_1 the second contribution to $\mathcal{W}_{t-a}$ is exponentially growing leading us to the condition $c_2 = 0$. As a consequence we arrive at

$$\mathcal{W}_{t-a}(\xi_1, \xi_2) = 0 \quad \text{for } k_0 < |k|. \tag{31}$$

This argument is similar to the one used in [18] to show that the wave function outside of an infinitely high potential must vanish.

4.1.2 Allowed domain. We now solve the differential equation

$$\left[\frac{\partial^2}{\partial \xi_1^2} + q^2(k)\right]\mathcal{W}_{t-a}(\xi_1, k) = 0 \tag{32}$$

for $|k| < k_0$ with

$$q^2(k) \equiv 4\frac{k_0^2 - k^2}{k^2}. \tag{33}$$

Since q^2 is positive we have the obvious solution

$$\mathcal{W}_{t-a} = g(k)\cos\left[q(k)\left(xk_y - yk_x\right) + \Phi(k)\right] \equiv g\cos S. \tag{34}$$

Since we are dealing with a differential equation of second order, two constants of integration, that is amplitude g and the phase Φ, appear. Moreover, the fact that we face a *partial* differential equation these quantities are constant with respect to the variable ξ_1 but still functions of k^2, that is $g = g(k)$ and $\Phi = \Phi(k)$. They are determined by the Weyl–Wigner eigenvalue equation of angular momentum.

We can combine the case $|k| < k_0$ and $k_0 < |k|$ by introducing the Heaviside step function Θ which yields

$$W_E(\mathbf{r}, \mathbf{k}) = \Theta\left(k_0^2 - k^2\right)g(k)\cos S(x, y, k). \tag{35}$$

4.2 *Angular momentum equation*

We now address the Weyl–Wigner eigenvalue equation of angular momentum, equation (21). For this purpose we substitute the ansatz, equation (35), into equation (21). Since the Heaviside step function Θ, the amplitude g and the phase S depend on k, the differentiations with respect to k_x and k_y are rather complicated and we perform them in detail in Appendix A. As a result we arrive at

$$\begin{aligned}&-\kappa\Theta g\cos S+\frac{1}{4}\Theta gqk\frac{\mathrm{d}\Phi}{\mathrm{d}k}\cos S+\frac{1}{2}\Theta gq\sin S\\&-\frac{1}{2}k^2\delta gq\sin S+\frac{1}{4}\Theta\frac{\mathrm{d}}{\mathrm{d}k}(gq)k\sin S=0.\end{aligned}\tag{36}$$

We now combine the terms proportional to cosine and sine which leads to the two equations

$$\frac{\mathrm{d}\Phi}{\mathrm{d}k}=\frac{4\kappa}{kq}=\frac{2\kappa}{(k_0^2-k^2)^{1/2}}\tag{37}$$

and

$$\Theta gq-k^2\delta gq+\frac{1}{2}\Theta\frac{\mathrm{d}}{\mathrm{d}k}(gq)k=0.\tag{38}$$

From the first equation we find

$$\Phi(k)=2\kappa\ \arcsin\left(\frac{k}{k_0}\right)+\Phi_0,\tag{39}$$

where $\Phi_0\equiv\Phi(k=0)$ is the phase for $k=0$.

We now address the second equation and consider first the case $k\neq k_0$. As a consequence the contribution from the δ-function vanishes and we find

$$gq+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}k}(gq)k=0\tag{40}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}k}(gq)=-2\frac{1}{k}gq\tag{41}$$

with the solution

$$gq=\mathcal{N}\frac{1}{k^2},\tag{42}$$

where $\mathcal{N}$ is a constant.

As a result the Wigner function of an energy eigenstate in two space dimensions reads

$$W_E(x,y,k)=\Theta\big(k_0^2-k^2\big)\mathcal{N}\frac{\cos\big[2\big([(k_0^2-k^2)^{1/2}/k]\big(xk_y-yk_x\big)+\kappa\arcsin(k/k_0)\big)+\Phi_0\big]}{2k(k_0^2-k^2)^{1/2}}.\tag{43}$$

5. Determination of the constants

The expression, equation (43), still contains the three unknown quantities $\mathcal{N}$, κ and Φ_0. We now determine these parameters from physical conditions and relate them to the normalization, the quantization of angular momentum and the time-reversal symmetry.

5.1 *Quantization of angular momentum*

We start our considerations with equation (36) in the limit of $k = k_0$. In this case the contributions due to the Heaviside step functions cannot compensate the infinity due to the δ-function. The only way to eliminate this term is to choose the phase Φ_0 such that the sine function vanishes at $k = k_0$. This condition is going to provide us with the quantization condition of angular momentum, that is, κ being an integer. Indeed, we find from the condition

$$\sin\, S(k_0) = \sin\left(q(k_0)\left(xk_y - yk_x\right) + \Phi(k_0)\right) = 0 \tag{44}$$

with the relations $q(k_0) = 0$ and $\Phi(k_0) = \kappa\pi + \Phi_0$ following from the expressions, equations (33) and (39), for q and Φ the condition

$$\kappa\pi + \Phi_0 = l\pi, \tag{45}$$

where l is an integer.

5.2 *Time-reversal symmetry*

Needless to say equation (45) is only one equation for the two unknown parameters κ and Φ_0. Therefore, we need to find a second condition which follows from the Weyl–Wigner eigenvalue equation of angular momentum, equation (21). For this purpose we replace $\mathbf{k}$ by $-\mathbf{k}$ in equation (21). Since the wave vector coordinates appear in every term in a linear way we deduce the symmetry relation

$$W_{k_0^2,\kappa}(x, y, -\mathbf{k}) = W_{k_0^2,-\kappa}(x, y, \mathbf{k}) \tag{46}$$

relating the Wigner function of eigenvalue $-\kappa$ and arguments $(x, y, \mathbf{k})$ to the Wigner function of eigenvalue κ and arguments $(x, y, -\mathbf{k})$. This relation is the phase space analogue of the symmetry relation

$$\psi_m^*(x, y) = \exp\left(\mathrm{i}\delta_m\right)\psi_{-m}(x, y) \tag{47}$$

following from the angular momentum eigenvalue equation, equation (8), by complex conjugation. The value of the arbitrary phase factor δ_m depends on convention.

The physical symmetry that we have deduced here is the fundamental time-reversal symmetry [19]. In polar coordinates r and φ this transformation leaves r invariant and replaces φ by $-\varphi$. As a result the energy wave function, equation (1), displays the symmetry relation

$$\psi_m^*(x, y) = N J_m(k_0 r)\exp\left(-\mathrm{i}m\varphi\right) = (-1)^m NJ_{-m}(k_0 r)\exp\left(-\mathrm{i}m\varphi\right) = (-1)^m\psi_{-m}(x, y). \tag{48}$$

Here we have used the identity [12]

$$(-1)^m J_m(z) = J_{-m}(z). \tag{49}$$

When we substitute the expression, equation (43), for the Wigner function into the symmetry relation, equation (46), we find with the help of the identity

$$\cos\left[-q(k)\left(xk_y - yk_x\right) + \Phi(k)\right] = \cos\left[q(k)\left(xk_y - yk_x\right) - 2\kappa \arcsin\left(\frac{k}{k_0}\right) - \Phi_0\right] \tag{50}$$

the condition

$$\begin{aligned}\cos&\left[q(k)\left(xk_y - yk_x\right) - 2\kappa \arcsin\left(\frac{k}{k_0}\right) - \Phi_0\right]\\ &= \cos\left[q(k)\left(xk_y - yk_x\right) - 2\kappa \arcsin\left(\frac{k}{k_0}\right) + \Phi_0\right],\end{aligned} \tag{51}$$

which can only be satisfied if

$$2\Phi_0 = 2n\pi, \tag{52}$$

where n is an integer.

5.3 *Combining both conditions*

We are now in the position to determine κ and Φ_0. For this purpose we combine equations (45) and (52) which yields

$$\kappa\pi + n\pi = l\pi \tag{53}$$

or

$$\kappa = l - n \equiv m. \tag{54}$$

Therefore, the Wigner function of an energy eigenstate in two space dimensions reads

$$\begin{aligned}&W_E(x, y, k)\\ &\quad = \Theta\left(k_0^2 - k^2\right)\mathcal{N}\frac{\cos\left[2\left(\left[(k_0^2 - k^2)^{1/2}/k\right]\left(xk_y - yk_x\right) + m \arcsin\left(k/k_0\right)\right) + m\pi\right]}{2k(k_0^2 - k^2)^{1/2}}.\end{aligned} \tag{55}$$

Here we have chosen n to be equal to m.

5.4 *New variables*

We now simplify the expression for the Wigner function and introduce the representation

$$\left(xk_y - yk_x\right) \equiv rk\sin\theta \tag{56}$$

of the angular momentum variable where θ denotes the angle between $\mathbf{k}$ and $\mathbf{r}$. We recognize that the Wigner function does not depend on the angle ϕ indicating the angle of the centre of $\mathbf{k}$ and $\mathbf{r}$ with respect to the x axis. As a consequence the volume element $\mathrm{d}V$ of the four-dimensional phase space takes the form

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}k_x\,\mathrm{d}k_y = \mathrm{d}r\,r\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}k\,k. \tag{57}$$

Moreover, we introduce the new variable

$$\zeta \equiv \arcsin\frac{k}{k_0} + \frac{\pi}{2}. \tag{58}$$

The domain $|k| \le k_0$ transforms into the region $0 \le \zeta \le \pi$.

With the help of the relation

$$\frac{\mathrm{d}\zeta}{\mathrm{d}k} = \frac{1}{(k_0^2 - k^2)^{1/2}}, \tag{59}$$

the quasi-probability $W_E \mathrm{d}V$ takes the form

$$W_E(x, y, k_x, k_y)\,\mathrm{d}V = \mathcal{W}_E(r, \zeta, \theta)\,\mathrm{d}r\,r\,\mathrm{d}\theta\,\mathrm{d}\zeta\,\mathrm{d}\phi \tag{60}$$

with

$$\mathcal{W}_E(r, \zeta, \theta) = \frac{1}{2}\mathcal{N}\cos[2k_0 r\sin\theta\sin\zeta + 2m\zeta]. \tag{61}$$

5.5 *Normalization*

The energy eigenfunctions of a free particle are normalized with respect to a Dirac δ-function, that is

$$\langle E|E'\rangle = \delta(E - E'). \tag{62}$$

As a consequence the integral of the Wigner over all phase space is infinite. Nevertheless, the Wigner function integrated over the wave vector must lead us to the position distribution of the energy eigenstate, that is

$$\int \mathrm{d}^2k\, W_E(\mathbf{r}, \mathbf{k}) = \left|\psi_m(r, \varphi)\right|^2 = \frac{M}{2\pi\hbar^2} J_m^2(k_0 r). \tag{63}$$

This relation allows us to determine $\mathcal{N}$. Indeed, we find from the identity

$$\int_0^\infty \mathrm{d}k\,k \int_{-\pi}^{\pi} \mathrm{d}\theta\, W_E = \int_0^{k_0} \mathrm{d}k\,k \int_{-\pi}^{\pi} \mathrm{d}\theta\, W_E = \int_0^{\pi} \mathrm{d}\zeta \int_{-\pi}^{\pi} \mathrm{d}\theta\, \mathcal{W}_E \tag{64}$$

together with the integral representation

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \mathrm{d}\zeta\,\cos(n\zeta - z\sin\zeta) \tag{65}$$

of the Bessel function and the integral relation

$$J_n^2(z) = \frac{1}{\pi}\int_0^{\pi} \mathrm{d}\theta\, J_{2n}(2z\sin\theta), \tag{66}$$

the condition

$$\pi^2 \mathcal{N} J_m^2(k_0 r) = \frac{M}{2\pi\hbar^2} J_m^2(k_0 r) \tag{67}$$

or

$$\mathcal{N} = \frac{M}{2\pi^3\hbar^2}. \tag{68}$$

However, this technique is not quite fair since here we have used explicitly the wave function. To avoid this complication we recall the completeness relation

$$\int_0^\infty \mathrm{d}E\,|E\rangle\langle E| = \hat{\mathbb{1}} \tag{69}$$

of the energy eigenstates which translates into the identity

$$\begin{aligned}\int_0^\infty \mathrm{d}E\, W_E &= \frac{1}{(2\pi)^2}\int \mathrm{d}^2\xi \langle \mathbf{x} - \tfrac{1}{2}\xi|\hat{\mathbb{1}}|\mathbf{x} + \tfrac{1}{2}\xi\rangle\, \exp(-\mathrm{i}\boldsymbol{\xi}\cdot\mathbf{k}) \\ &= \frac{1}{(2\pi)^2}\int \mathrm{d}^2\xi\, \delta(\xi)\, \exp(-\mathrm{i}\boldsymbol{\xi}\cdot\mathbf{k}) = \frac{1}{(2\pi)^2}.\end{aligned} \tag{70}$$

The integration over all energies translates into an integration over all wave vectors k_0 and summation over all angular momenta quantum numbers m. From equation (5) we find

$$\frac{\hbar^2}{M}\int_0^\infty \mathrm{d}k_0\, k_0 \sum_{m=-\infty}^{\infty} W_{k_0,m} = \frac{1}{(2\pi)^2}. \tag{71}$$

When we substitute the expression, equation (43), for the Wigner function into the sum

$$\begin{aligned}\sigma \equiv \sum_{m=-\infty}^{\infty} W_{k_0^2,m} &= \mathcal{N}\frac{\exp\left[\mathrm{i}2(k_0^2 - k^2)^{1/2} r \sin\theta\right]}{4k(k_0^2 - k^2)^{1/2}} \\ &\times \sum_{m=-\infty}^{\infty} \exp\left[\mathrm{i}m\left(2\arcsin\left(\frac{k}{k_0}\right) + \pi\right)\right] + c.c.\end{aligned} \tag{72}$$

and make use of the relation

$$\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} \exp[\mathrm{i}mf(\beta)] = \delta(f(\beta)) = \left|\frac{\partial f(\beta)}{\partial\beta}\right|_{\beta=\beta_0}^{-1} \delta(\beta - \beta_0), \tag{73}$$

the sum σ reduces to

$$\sigma = \mathcal{N}\frac{\pi}{2k_0}\delta(k_0 - k), \tag{74}$$

which allows us to perform the remaining integration

$$\frac{\hbar^2}{M}\int_0^\infty \mathrm{d}k_0\, k_0 \sigma = \mathcal{N}\frac{\hbar^2\pi}{2M} = \frac{1}{(2\pi)^2} \tag{75}$$

leading to

$$\mathcal{N} = \frac{M}{2\pi^3 \hbar^2}, \tag{76}$$

in complete agreement with the derivation from the position distribution.

6. Summary

In the present paper we have derived the Wigner function of an energy eigenstate of a free particle in two space dimensions. Here we have not started from the wave function and calculated the standard integral but have rather solved partial differential equations in phase space. We have obtained these differential equations from the Weyl–Wigner correspondence of the Schrödinger eigenvalue equations of energy and angular momentum in two space dimensions. The expression for the Wigner function is remarkable for three reasons: (i) it vanishes exactly for wave vectors $k_0 < |k|$, (ii) it has a square root singularity at $k = k_0$ and a pole at $k = 0$ and (iii) in appropriate coordinates the Wigner function together with the volume element is an elementary cosine wave. In this representation the coordinates and the wave vector only enter through the combinations of variables $k_0 r$, the angle θ between $\mathbf{r}$ and $\mathbf{k}$ and the absolute value k of $\mathbf{k}$ expressed by the angle ζ as $k = k_0 \cos \zeta$. Moreover, we have derived the quantization of angular momentum from phase space. It emerges through an intricate interplay between the time-reversal symmetry of the state and the fact that the Wigner quasi-probability function has to remain finite.

We are now in the position to answer the question raised in the introduction concerning the manifestations of the attractive or repulsive interference forces in the Wigner function. For this purpose we could represent the Wigner function $\mathcal{W}_E(r, \zeta, \theta)$ for fixed $k_0 r$ in its dependence on ζ and θ. Here the dependence of the negative domains of $\mathcal{W}_E$ on m is of interest. However, this discussion goes beyond the scope of the present paper.

Another interesting extension of our analysis is to calculate the Wigner function $W_{E,E'}$ for transitions between the energy eigenstates of energy E and E'. Here the emergence of the quantization condition is of interest. Unfortunately we also have to postpone this analysis to a future paper.

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Appendix A: differentiations

In this appendix we substitute the ansatz

$$W_E = \Theta\left(k_0^2 - k^2\right)g(k)\,\cos\left[q(k)\left(xk_y - yk_x\right) + \Phi(k)\right] \equiv \Theta g\,\cos\,S \tag{A1}$$

for the Wigner function into the differential equation

$$\left[xk_y - yk_x - \kappa + \frac{1}{4}O\right]W_E = 0 \tag{A2}$$

with the operator

$$O \equiv \frac{\partial^2}{\partial k_x \partial y} - \frac{\partial^2}{\partial k_y \partial x} \tag{A3}$$

following from the eigenvalue equation of angular momentum. In order to simplify the notation we sometimes suppress the arguments of the functions.

We first note the identities

$$\frac{\partial}{\partial y}W_E = \Theta gq \sin S\, k_x \equiv f k_x \tag{A4}$$

and

$$\frac{\partial}{\partial x}W_E = -\Theta gq \sin S\, k_y \equiv f k_y. \tag{A5}$$

We now differentiate these quantities with respect to k_x and k_y which yields

$$OW_E = \frac{\partial}{\partial k_x}(f k_x) + \frac{\partial}{\partial k_y}\left(f k_y\right) = 2f + k_x\frac{\partial}{\partial k_x}f + k_y\frac{\partial}{\partial k_y}f \tag{A6}$$

and

$$\frac{\partial}{\partial k_x}f = \delta\left(k_0^2 - k^2\right)(-2k_x)gq\,\sin\,S + \Theta\frac{\mathrm{d}}{\mathrm{d}k}(gq)\frac{k_x}{k}\,\sin\,S + \Theta gq\,\cos\,S\frac{\partial S}{\partial k_x}. \tag{A7}$$

Here we have used the relation

$$\frac{\partial}{\partial k_j}h(k) = \frac{\mathrm{d}k}{\mathrm{d}k_j}\frac{\mathrm{d}h}{\mathrm{d}k} = \frac{k_j}{k}\frac{\mathrm{d}h}{\mathrm{d}k} \tag{A8}$$

for a function $h = h(k)$ which depends on k only.

With the help of the explicit form, equation (A1), of S we derive

$$\frac{\partial S}{\partial k_x} = \frac{\mathrm{d}q}{\mathrm{d}k}\frac{k_x}{k}\left(xk_y - yk_x\right) - qy + \frac{\mathrm{d}\Phi}{\mathrm{d}k}\frac{k_x}{k} \tag{A9}$$

and thus

$$\frac{\partial}{\partial k_x} f = -2k_x \delta g q \sin S + \Theta \frac{\mathrm{d}}{\mathrm{d}k}(gq)\frac{k_x}{k}\sin S + \Theta g q \cos S\left\{\frac{\mathrm{d}q}{\mathrm{d}k}\frac{k_x}{k}\left(xk_y - yk_x\right) - qy + \frac{\mathrm{d}\Phi}{\mathrm{d}k}\frac{k_x}{k}\right\}. \tag{A10}$$

Likewise we find

$$\frac{\partial}{\partial k_y} f = -2k_y \delta g q \sin S + \Theta \frac{\mathrm{d}}{\mathrm{d}k}(gq)\frac{k_y}{k}\sin S + \Theta g q \cos S\left\{\frac{\mathrm{d}q}{\mathrm{d}k}\frac{k_y}{k}\left(xk_y - yk_x\right) + qx + \frac{\mathrm{d}\Phi}{\mathrm{d}k}\frac{k_y}{k}\right\} \tag{A11}$$

which yields

$$k_x\frac{\partial}{\partial k_x} f + k_y\frac{\partial}{\partial k_y} f = -2k^2 \delta g q \sin S + \Theta \frac{\mathrm{d}}{\mathrm{d}k}(gq)k\sin S + \Theta g q \cos S\left\{\left(\frac{\mathrm{d}q}{\mathrm{d}k}k + q\right)\left(xk_y - yk_x\right) + \frac{\mathrm{d}\Phi}{\mathrm{d}k}k\right\}. \tag{A12}$$

From the definition

$$q(k) = 2\frac{(k_0^2 - k^2)^{1/2}}{k} \tag{A13}$$

of q we derive the identity

$$\frac{\mathrm{d}q}{\mathrm{d}k}k + q = -\frac{4}{q} \tag{A14}$$

and arrive at

$$k_x\frac{\partial}{\partial k_x} f + k_y\frac{\partial}{\partial k_y} f = -2k^2 \delta g q \sin S + \Theta \frac{\mathrm{d}}{\mathrm{d}k}(gq)k\sin S - 4\left(xk_y - yk_x\right)W_E + \Theta g q k \frac{\mathrm{d}\Phi}{\mathrm{d}k}\cos S.$$

We substitute this expression into the formula, equation (A 6), for $O\,W_E$ and arrive at

$$O\,W_E = -4\left(xk_y - yk_x\right)W_E + 2\Theta g q \sin S - 2k^2\delta g q \sin S + \Theta\frac{\mathrm{d}}{\mathrm{d}k}(gq)k\sin S + \Theta g q k\frac{\mathrm{d}\Phi}{\mathrm{d}k}\cos S \tag{A15}$$

and equation (A 2) takes the form

$$-\kappa\Theta g\cos S + \frac{1}{4}\Theta g q k\frac{\mathrm{d}\Phi}{\mathrm{d}k}\cos S + \frac{1}{2}\Theta g q \sin S - \frac{1}{2}k^2\delta g q\sin S + \frac{1}{4}\Theta\frac{\mathrm{d}}{\mathrm{d}k}(gq)k\sin S = 0. \tag{A16}$$

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