The Heisenberg model on the triangular lattice, again

Federico Becca

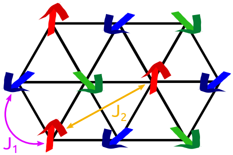
Quantum Spin Liquids 2025: Experimental Enigmas & Theoretical Challenges
Budapest, October 2025

UNIVERSITÀ
DEGLI STUDI DI TRIESTE

S. Budaraju, A. Parola, Y. Iqbal, FB, and D. Poilblanc, Phys. Rev. B **111**, 125150 (2025)

S. Budaraju *et al.*, in preparation

$$\mathcal{H} = J_1 \sum_{\langle R,R' \rangle} \mathbf{S}_R \cdot \mathbf{S}_{R'} + J_2 \sum_{\langle\langle R,R' \rangle\rangle} \mathbf{S}_R \cdot \mathbf{S}_{R'}$$



- **Magnetic order** settles down in weakly frustrated cases

  Spontaneous symmetry breaking of the SU(2) spin symmetry $\langle \mathbf{S}_R \rangle \neq 0$

- Elementary excitations are **gapless magnons**, i.e., $S = 1$ objects

  Goldstone modes of the broken symmetry

- The low-energy theory is described by **semi-classical** approaches

  Holstein-Primakov bosons and spin-wave approximation

T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1940)

Looking for a magnetically disordered ground state

- Many theoretical suggestions since P.W. Anderson (1973)

P.W. Anderson, Mater. Res. Bull. **8**, 153 (1973)

P. Fazekas and P.W. Anderson, Phil. Mag. **30**, 423 (1974)

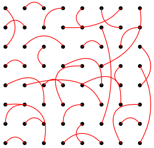
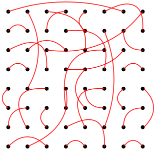
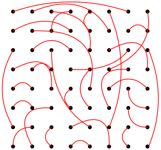
"Resonating valence-bond" (RVB) states

Idea: the best state for two spin-1/2 spins is a valence bond (a spin singlet):

$$|VB\rangle_{\mathbf{R},\mathbf{R}'} = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_{\mathbf{R}} |\downarrow\rangle_{\mathbf{R}'} - |\downarrow\rangle_{\mathbf{R}} |\uparrow\rangle_{\mathbf{R}'} \right)$$

Every spin of the lattice is coupled to a partner

Then, take a superposition of different valence bond configurations

$\Psi =$  $+$  $+$  $+ \dots$

- **The low-energy properties are described by gauge theories**

Matter fields are fractional $S = 1/2$ (spinons)

Gauge fields are emerging from the fractionalization approach

G. Baskaran and P.W. Anderson, Phys. Rev. B **37**, 580 (1988)

N. Read and S. Sachdev, Phys. Rev. Lett. **66**, 1773 (1991)

X.-G. Wen, *Quantum Field Theory of Many-Body Systems* (Oxford, 2004)

Z_2 gauge fields + spinons $\Longrightarrow$ stable deconfined phase

N. Read and S. Sachdev, Phys. Rev. Lett. 66, 1773 (1991)

T. Senthil and M.P. Fisher, Phys. Rev. B 62, 7850 (2000)

A. Kitaev, Ann. Phys. **321**, 2 (2006)

U(1) gauge fields + spinons $\Longrightarrow$???

A.M. Polyakov, Nucl. Phys. B **120**, 429 (1977)

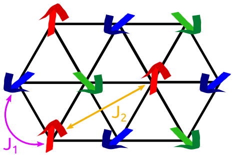
N. Read and Sachdev, Phys. Rev. Lett. **62**, 1694 (1989)

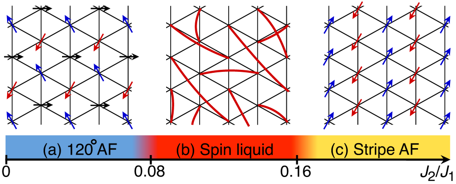
On frustrated lattices, a deconfined phase could be possible

X.-Y. Song, C. Wang, A. Vishwanath, and Y.-C. He, Nat. Comm. **10**, 4254 (2019)

A. Wietek, S. Capponi, and A.M. Läuchli, Phys. Rev. X **14**, 021010 (2024)

$$\mathcal{H} = J_1 \sum_{\langle R,R' \rangle} \mathbf{S}_R \cdot \mathbf{S}_{R'} + J_2 \sum_{\langle\langle R,R' \rangle\rangle} \mathbf{S}_R \cdot \mathbf{S}_{R'}$$





- Coplanar antiferromagnetic (120°) order for $J_2/J_1 \lesssim 0.08$
- Spin liquid (presumably gapless) for $0.08 \lesssim J_2/J_1 \lesssim 0.16$
- Collinear antiferromagnet $0.16 \lesssim J_2/J_1 <$???

Z. Zhu and S.R. White, Phys. Rev. B **92**, 041105 (2015)

W.-J. Hu, S.-S. Gong, W. Zhu, and D.N. Sheng, Phys. Rev. B **92**, 140403 (2015)

Y. Iqbal, W.-J. Hu, R. Thomale, D. Poilblanc, and F. Becca, Phys. Rev. B **93**, 144411 (2016)

S. Hu, W. Zhu, S. Eggert, and Y.-C. He, Phys. Rev. Lett. **123**, 207203 (2019)

- Consider the spin-1/2 Heisenberg model on a generic lattice

$$\mathcal{H} = \sum_{R,R'} J_{R,R'} \mathbf{S}_R \cdot \mathbf{S}_{R'}$$

- A faithful representation of spin-1/2 is given by

$$\boxed{S^a_R = \frac{1}{2} c^\dagger_{R,\alpha} \sigma^a_{\alpha,\beta} c_{R,\beta}}$$

SU(2) gauge redundancy
e.g., $c_{R,\beta} \to e^{i\theta_R} c_{R,\beta}$

- The spin model is transformed into a purely interacting fermionic system

$$\mathcal{H} = \sum_{R,R'} J_{R,R'} \sum_{\sigma,\sigma'} \left(\sigma\sigma' c^\dagger_{R,\sigma} c_{R,\sigma} c^\dagger_{R',\sigma'} c_{R',\sigma'} + \frac{1}{2} \delta_{\sigma',\bar{\sigma}} c^\dagger_{R,\sigma} c_{R,\sigma'} c^\dagger_{R',\sigma'} c_{R',\sigma} \right)$$

- One spin per site $\to$ we must impose the constraint

$$\boxed{c^\dagger_{i,\uparrow} c_{i,\uparrow} + c^\dagger_{i,\downarrow} c_{i,\downarrow} = 1}$$

- The SU(2) symmetric "mean-field" approximation gives a BCS-like form

$$\mathcal{H}_0 = \sum_{R,R',\sigma} t_{R,R'} c^\dagger_{R,\sigma} c_{R',\sigma} + \sum_{R,R'} \Delta_{R,R'} c^\dagger_{R,\uparrow} c^\dagger_{R',\downarrow} + h.c.$$

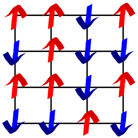
$\{t_{R,R'}\}$ and $\{\Delta_{R,R'}\}$ define the "mean-field" Ansatz $\Longrightarrow$ BCS spectrum $\{\epsilon_\alpha\}$

The constraint is no longer satisfied locally (only on average)

- The constraint can be inserted by the Gutzwiller projector → **RVB**

$$\boxed{|\Psi_0\rangle = \mathcal{P}_G |\Phi_0\rangle}$$

$$\mathcal{P}_G = \prod_R (n_{R,\uparrow} - n_{R,\downarrow})^2$$



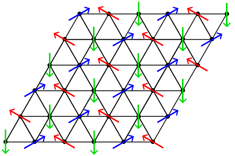
- The exact projection can be treated within the variational Monte Carlo approach

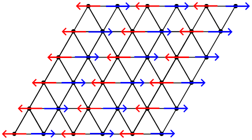
F. Becca and S. Sorella, *Quantum Monte Carlo Approaches for Correlated Systems*

- Magnetic states [breaking SU(2)] can be also constructed

$$\mathcal{H}_0 = \sum_{R,R',\sigma} t_{R,R'} c^\dagger_{R,\sigma} c_{R',\sigma} + \sum_{R} \mathbf{h}_R \cdot \mathbf{S}_R$$

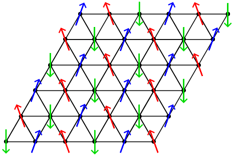
Magnetic state for $J_2/J_1 \lesssim 0.08$ and $J_2/J_1 \gtrsim 0.16$

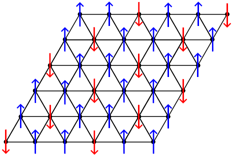




Y. Iqbal, W.-J. Hu, R. Thomale, D. Poilblanc, and F. Becca, Phys. Rev. B **93**, 144411 (2016)

Magnetic states in an external magnetic field





S. Budaraju, next talk after the coffee break

- Within the spin-liquid regime $0.07 \lesssim J_2/J_1 \lesssim 0.16$

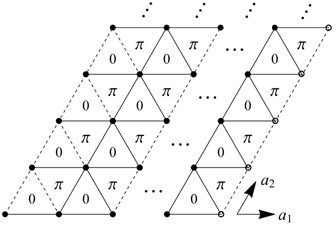
  Only hopping with non-trivial sign structure

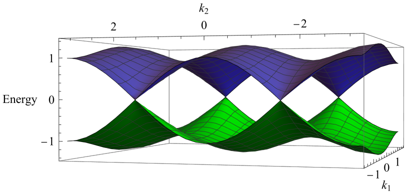
  Two Dirac points in the reduced Brillouin zone ($N_f = 4$ gapless fermions)

$$\mathcal{H}_0 = \sum_{R,R',\sigma} t_{R,R'} c^\dagger_{R,\sigma} c_{R',\sigma}$$

- **Accurate ground-state energies are obtained (compared to other methods)**

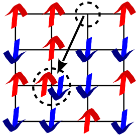
Y. Iqbal, W.-J. Hu, R. Thomale, D. Poilblanc, and F. Becca, Phys. Rev. B **93**, 144411 (2016)





- For each momentum q a set of (two-spinon) states is defined

$$\boxed{|q,R\rangle = \mathcal{P}_G \frac{1}{\sqrt{L}} \sum_{R'} e^{iqR'} (c^\dagger_{R+R',\uparrow} c_{R',\uparrow} - c^\dagger_{R+R',\downarrow} c_{R',\downarrow}) |\Phi_0\rangle}$$



- The spin Hamiltonian is diagonalized within this (non-orthogonal) basis set

$$\boxed{\sum_{R'} H^q_{R,R'} A^{n,q}_{R'} = E^q_n \sum_{R'} O^q_{R,R'} A^{n,q}_{R'}}$$

- The Matrix elements are computed within standard variational Monte Carlo

T. Li and F. Yang, Phys. Rev. B **81**, 214509 (2010)

- The generic "eigenstate" of the Hamiltonian is

$$\boxed{|\Psi_n^q\rangle = \sum_R A_R^{n,q} |q, R\rangle}$$

If $A_R^{n,q} = \delta_{R,0}$, on-site particle-hole excitations in $|q, R\rangle$

We obtain the the single-mode approximation (magnon)

$$|\Psi_n^q\rangle = S_q^z |\Psi_0\rangle$$

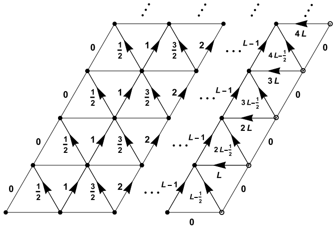
In general, $A_R^{n,q} \neq \delta_{R,0}$, non-local particle-hole excitations in $|q, R\rangle$

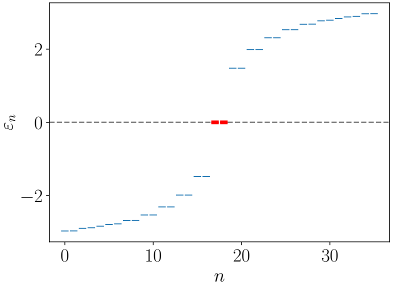
- The dynamical structure factor is approximated by

$$\boxed{S^z(q,\omega) = \sum_n \left| \sum_R (A_R^{n,q})^* O_{R,0}^q \right|^2 \delta(\omega - E_n^q + E_0)}$$

At most L states for each momentum q

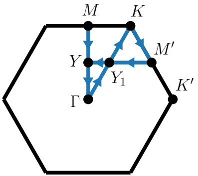
- **Low-energy gauge fluctuations change the flux pattern $\Longrightarrow$ U(1) symmetry**
- A $\pm 2\pi$ flux on top of the Dirac state ($Q = 1$ monopole)

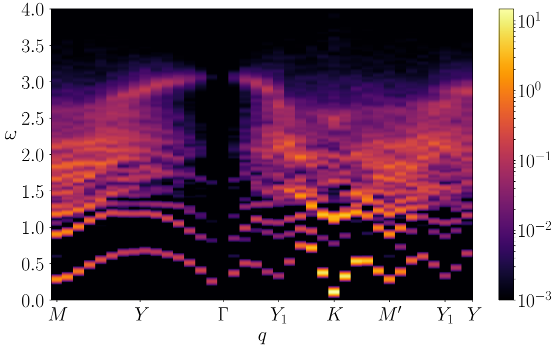




- For each choice $\phi = \pm 2\pi$, 4 orbitals to be filled with 2 fermions
  6 singlets and 2 triplets
- $Q > 1$ monopoles can be also constructed with $\pm 2\pi Q$ fluxes

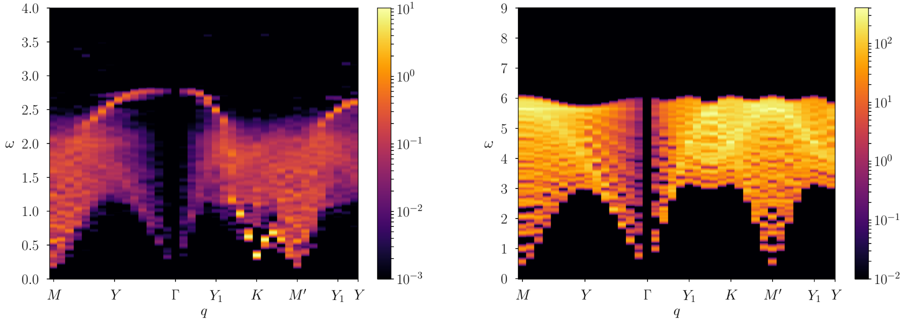
- $J_1 - J_2$ model with $J_2/J_1 = 0.07$





- **Low-energy excitations at K point (magnon)**
- **Low-energy excitations at M point (bilinear of spinons)**

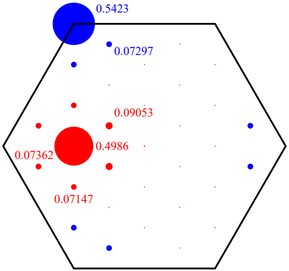
- $J_1 - J_2$ model with $J_2/J_1 = 0.125$



- **Low-energy excitations at K point (monopole?)**
- **Low-energy excitations at M point (bilinear of spinons)**

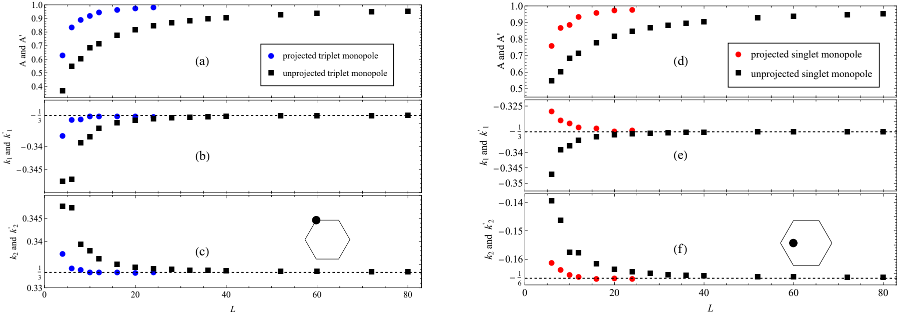
- **Monopole states break translational symmetries**
- On the 6×6 cluster, symmetries can be imposed (brutally)

$$\mathcal{N}_k^2 = \langle \Psi | \mathcal{P}_k^2 | \Psi \rangle \qquad \sum_k \mathcal{N}_k^2 = 1$$



- **Triplet monopole at k** $= (-2\pi/3, 2\pi/\sqrt{3})$
- **Singlet monopoles at k** $= (-2\pi/3, 0)$ [**k** $= (\pi/3, \pm\pi/\sqrt{3})$]

- **Large overlaps (**0.69 **and** 0.62**) with exact low-energy states**
- **Large overlap (**0.67 **and** 0.64**) between monopole and two-spinon states!**

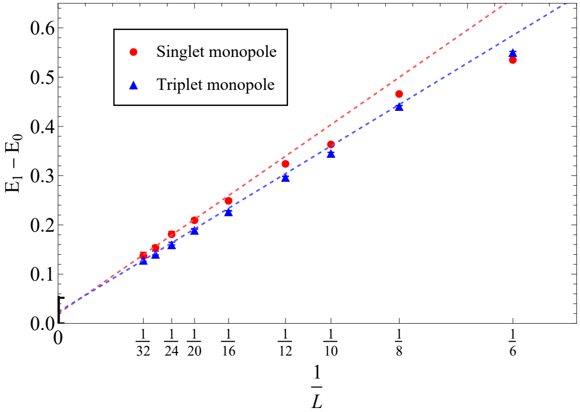


- Unprojected wave function $$\frac{\langle \Phi | \mathcal{G}^m_{T_j} T_j | \Phi \rangle}{\langle \Phi | \Phi \rangle} = A e^{i k_j}$$
- Projected wave function $$\frac{\langle \Psi | T_j | \Psi \rangle}{\langle \Psi | \Psi \rangle} = A' e^{i k'_j}$$

X.-Y. Song, C. Wang, A. Vishwanath, and Y.-C. He, Nat. Comm. **10**, 4254 (2019)

S. Budaraju, A. Parola, Y. Iqbal, FB, and D. Poilblanc, Phys. Rev. B **111**, 125150 (2025)

- The $Q = 1$ Monopole state is gapless (in the thermodynamic limit)



- **The monopole excitations are gapless (as in the large-N_f limit)**

S. Budaraju, A. Parola, Y. Iqbal, FB, and D. Poilblanc, Phys. Rev. B **111**, 125150 (2025)

- **On the conformal sphere (large-N_f)**

$$E_Q - E_0 = \frac{\Delta_Q^0 N_f + \Delta_Q^1}{R}$$

with $\Delta_Q^0 \propto Q^{3/2}$

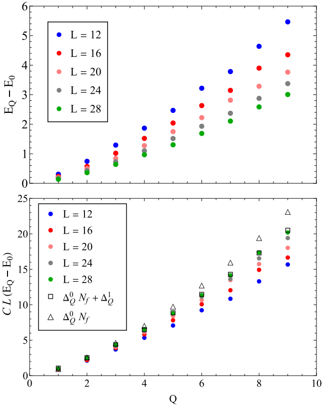
V. Borokhov, A. Kapustin, and X. Wu, J. High Energy Phys. **11** (2002) 049

E. Dyer, M. Mezei, and S.S. Pufu, arXiv:1309.1160

E. Dupuis, R. Boyack, and W. Witczak-Krempa, Phys. Rev. X **12**, 031012 (2022)

- **On the $L \times L$ torus ($N_f = 4$)**

$$E_Q - E_0 \propto \frac{Q^{3/2}}{L}$$

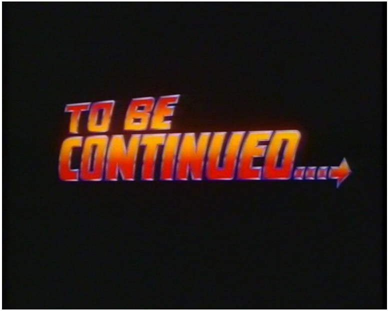


$$\mathcal{H} = J_1 \sum_{\langle R,R' \rangle} \mathbf{S}_R \cdot \mathbf{S}_{R'} + J_2 \sum_{\langle\langle R,R' \rangle\rangle} \mathbf{S}_R \cdot \mathbf{S}_{R'} + B \sum_R S_R^z$$

An old problem important for the future

A.V. Chubukov and D.I. Golosov, J. Phys.: Condens. Matter **3**, 69 (1991)

S. Budaraju *et al.*, in preparation



by S. Budaraju, next talk "Back to the Future"