Scaling and Crossovers in Diffusion Limited Aggregation

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We discuss the scaling of characteristic lengths in diffusion limited aggregation clusters in light of recent developments using conformal maps. We are led to the conjecture that the apparently anomalous scaling of lengths is due to one slow crossover. This is supported by an analytical argument for the scaling of the penetration depth of newly arrived random walkers, and by numerical evidence on the Laurent coefficients which uniquely determine each cluster. We find common crossover behavior for the squares of the characteristic lengths and the penetration depth of the form $N^{2/D}(\alpha + \beta N^{-\phi})$ with ϕ in the range -0.3 ± 0.1 suggesting that there is a single dominant correction to scaling.

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Diffusion limited aggregation (DLA) is a model for growth that was introduced in 1981 by Witten and Sander [1] and has been the subject of intensive scrutiny [2] ever since. Remarkable progress has been made in numerical treatments of the model and in applications to various physical problems. Nevertheless, a fundamental understanding of the most striking features of DLA clusters is still lacking. A very promising recent development is the method of iterated conformal maps introduced by Hastings and Levitov [3] (HL) and pursued by Davidovitch *et al.* [4]. Using the method of conformal maps we will try to clear up one of the major difficulties in the field, namely, the large corrections to scaling which lead to slow crossovers. This discussion has a very significant outcome: there has been serious doubt about whether DLA clusters are really fractal, based on the existence of characteristic lengths apparently scaling *differently* from the overall cluster radius. We show that these lengths cross over to scale in the same way as the radius and suggest that all of the various deviations from exact fractal scaling have the same source, with the same correction to scaling exponent. There is, in our view, no longer any reason to doubt that DLA clusters are fractal.

In the DLA model a nucleation center is fixed and random walkers are released from outside. When a walker comes in contact with the cluster it sticks. The process continues until N walkers have attached; in modern work N's of 10^6 are easy to attain. Provided lattice anisotropy is avoided, the clusters so produced seem to be self-similar with fractal dimension $D \approx 1.71$ in two dimensions and the probability of growth at a point on the surface appears to be multifractal.

The conformal map method uses the *Laplacian growth* [5,6] version of DLA. That is, we take the cluster to be a grounded conductor in the z plane, with a probability to grow at a point on its surface proportional to the charge there: $|\nabla V|$, where V is the potential with boundary conditions of unit flux at infinity and $V = 0$ on the surface. If we construct a complex potential such that $\mathrm{Re}[\Psi(z)] = V$ and define $w = e^{\Psi} = Z^{-1}(z)$, then $Z(w)$ is a conformal map which takes the exterior of the unit circle in the w plane to the exterior of the DLA cluster in the z plane, and is linear, $Z \sim r_0 w$, for large $|w|$. Thus $V \to \ln(|z|/r_0)$ at large $|z|$. The Laplace radius, r_0, is the radius of the grounded disk with the same capacitance (with respect to any distant reference point) as the cluster [4]. The growth probability is $1/|Z'|$. Intervals $d\theta$ on the unit circle in the w plane correspond to intervals of arc length ds with equal growth probability in the z plane.

To characterize the map we write

$$Z(w) = r_0 w + \sum_{k=1} A_k/w^k. \qquad (1)$$

The Laurent coefficients, A_k, are a useful parametrization of the map. In what follows we will assume that the Laurent expansion of Z has no constant term. This corresponds to moving the DLA cluster so that the center of charge is at the origin.

In the HL method [3,4] the A_k are produced directly. However, this method employs certain approximations, and also numerically difficult for large N because it is of order N^2, and as a practical matter is limited to $N \approx 10^4$. We observe that there is a way around these problems. Suppose we produce a DLA cluster by the conventional method with random walkers (which is much faster) and then freeze it at some N. Then by recording where M random walkers would attach to the cluster we have a set of points z_m. These are at angle $\theta_m \approx 2\pi(m/M)$ in the w plane, since we are sampling the charge, and equal increments of charge correspond to equal increments of θ. The A_k are the Fourier coefficients of the function $z(\theta_m)$. We should note that everything in this paper results from the existence of the conformal map, and the behavior of the A_k, not on the detailed method of generation proposed by HL.

Numerically we have found, using either the HL method [4] or the conventional method described above (cf. Fig. 2), that $r_0 \propto N^{1/D}$ and that for all but the first few k's, $\langle |A_k|^2 \rangle \propto N^{2/D}$. For $k \lesssim 4$ the A_k appear to scale with a smaller power of N, a behavior that we will interpret below as a crossover.

We will be concerned first with penetration depths of the random walkers, i.e., how far from the origin they land. This has been a matter of considerable interest for some time because the width of the distribution of this quantity ("the width of the growth zone") seems to increase more slowly with N than the mean radius [7]. This is a disturbing observation since a real fractal should have no length scale other than its overall size. Some authors [8] have maintained that what was observed was a slow crossover and that asymptotically all lengths scale together, but others [9] have pointed out that even very large clusters are consistent with an "infinite drift scenario" in which DLA is not a fractal at all in the asymptotic limit. The structure of conformal map theory gives us a very elegant way to discuss these matters.

We can define a penetration depth as follows: suppose we follow a field line from large $|z|$ to the surface of the cluster. The displacement of the end point of the line from where it would terminate on the equivalent disk may be written as $(r_\parallel + i r_\perp)w \equiv Z(w) - r_0 w$, where $r_\parallel$ is the radial displacement and $r_\perp$ is the transverse displacement; see Fig. 1. The spread of $r_\parallel$ defines a penetration depth $\xi_\parallel$:

$$\xi_\parallel^2 = \frac{1}{2\pi} \oint r_\parallel^2 \, d\theta \,. \tag{2}$$

We also consider the analogous quantity $\xi_\perp$:

$$\xi_\perp^2 = \frac{1}{2\pi} \oint r_\perp^2 \, d\theta \,. \tag{3}$$

Using Eq. (1) we have

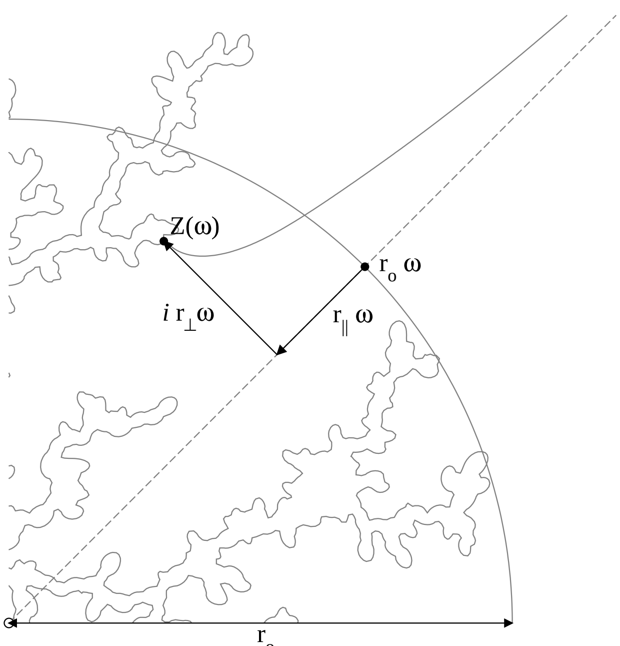

FIG. 1. The radial displacement $r_\parallel$ and the transverse displacement $r_\perp$. The origin is at the center of charge.

$$\xi_\parallel^2 = \xi_\perp^2 = \frac{1}{2} \sum_{k=1} |A_k|^2 . \tag{4}$$

This exact result has a useful geometric interpretation: penetration of a DLA cluster corresponds to landing inside one of the "fjords." Clearly $r_\perp$ is the azimuthal deviation of a walker from a straight path, and $r_\parallel$ is the penetration. Since they are the same, on average, the only way for DLA to have an anomalous scaling for the penetration depth is for the angular width of the fjords to be smaller and smaller as the cluster grows.

Now we can discuss the scaling of $\xi_\parallel$, the quantity of most interest, by analyzing the A_k and using Eq. (4). We have remarked above that the numerical evidence is that most of the $|A_k|$ scale as the radius. This is, in some sense obvious, since there is an elementary formula [4,10] that relates the area, S_N, of the image of the circle to the Laurent coefficients:

$$\pi r_0^2 = S_N + \pi \sum_{k=1} k |A_k|^2 . \tag{5}$$

The left hand side scales as $N^{2/D}$, whereas S_N is the sum of the areas of the deposited particles and is proportional to N. Thus, the leading behavior of the sum must also be $N^{2/D}$. The obvious way for this to occur is if each coefficient has this scaling, which would lead to $\xi_\parallel^2 \propto r_0^2$. We will show that this expectation is correct.

If we start with all the A_k small, then in the course of growth, they will not remain small. This is because Laplacian growth is subject to the Mullins-Sekerka instability [11]. In fact, the classic calculation of Ref. [11] was concerned with exactly this behavior: the Fourier coefficients of the wrinkling of the surface grow exponentially, with the growth rate proportional to k. The slowest growth is associated with the smallest k. However, we are concerned here with the limiting behavior of A_k, far out of the linear regime.

We can get some insight by making the following estimate for large k:

$$\begin{aligned} |A_k|^2 &= \frac{1}{4\pi^2} \int d\theta_1 \, d\theta_2 \, Z(e^{i\theta_1}) Z^*(e^{i\theta_2}) e^{ik(\theta_1 - \theta_2)} \\ &= \frac{1}{4\pi^2 k^2} \int d\theta_1 \, d\theta_2 \, \frac{dZ}{d\theta_1} \frac{dZ^*}{d\theta_2} \, e^{ik[\theta_1 - \theta_2]} \\ &\approx \frac{1}{4\pi^2 k^2} \sum_{\delta\theta = 1/k} |\Delta Z(e^{i\theta})|^2 , \end{aligned} \tag{6}$$

where the first step is an integration by parts, and the last line follows by taking the dominant behavior to be $\theta_1 \approx \theta_2$. The sum in the last equation can be evaluated in terms of the multifractal spectrum, $\tau(q)$. This follows from the partition function approach of Halsey *et al.* [12] where the spectrum is defined by the implicit equation:

$$\sum_{m=1}^{k} (P_m)^q (|\Delta Z_m|/r_0)^{-\tau(q)} = 1 \,. \tag{7}$$

Here the surface of the cluster should be thought of as being divided into k boxes. The charge in box m is P_m, and the box size is $|\Delta Z|$ for that box. In our case the charges are all equal to $1/k$, and we must put $\tau = -2$. Then we have

$$|A_k|^2 \sim (r_0^2/k^2) \sum [|\Delta Z|/r_0]^2 \sim r_0^2 k^{\hat{q}-2}, \quad (8)$$

where $\tau(\hat{q}) = -2$ [13]. We know that the function $\tau(q)$ is increasing and that $\tau(0) = -D$. Thus $\hat{q} < 0$ and we confirm that the sum in Eq. (5) is dominated by its first few terms, and that each scales as r_0^2. Furthermore, we have $\xi_\parallel^2 = \frac{1}{2} \sum_{k=1} |A_k|^2 \sim r_0^2$ asymptotically, and the sum converges rapidly.

Thus the asymptotic scaling behavior of the sum for the penetration length, Eq. (4), is the same as that of its terms and the sum is dominated by the first few. The crossover in the penetration depth must be associated with the crossover of *the first few Laurent coefficients* which, as we have seen, appear to behave differently from the prediction of Eq. (8). The first few coefficients (see Ref. [4]) represent the quadrupole, octupole, etc. moments of the cluster. We can guess that these have intrinsically slow dynamics (as in the linear regime) and thus cross over more slowly than higher moments. Any quantity associated with averages over the charge (the growth zone) should share this crossover. In fact, we conjecture that there is one kind of crossover of lengths with the same correction to scaling exponent.

We can verify this picture by numerical analysis of the penetration depth and other moments of the charge. We computed the Laurent coefficients r_0 and $A_1, \ldots, A_{10}$ for conventional DLA clusters using the method described earlier. For each cluster the number of points M sampling the harmonic measure was equal to the size of the cluster, but at least 10^5; this way the numerical error in the computed coefficients is around 10%, and the error in their ensemble average is much smaller. The size of the clusters range from 10^3 to 10^6, and for each given size an ensemble of 1000 was taken.

In Fig. 2(a) we show the penetration depth and compare its crossover to that of the squared absolute value of the Laurent coefficients. All of them seem to have asymptotic scaling of the form $r_0^2(\alpha + \beta N^{-\phi})$, where the exponent $\phi = 0.3 \pm 0.1$. The error in ϕ is estimated by comparing to Figs. 2(b) and 2(c) where the data are plotted against $N^{-0.2}$ and $N^{-0.4}$. The large N regime is shown because corrections to scaling are smaller and linearity of the plot is therefore more significant. The numerical results obtained by the HL method are close, especially for larger cluster sizes, to those shown in Fig. 2.

What we have shown in this paper is that we can unify many of the puzzling results on slow crossovers in DLA. The conformal map approach gave us a strong indication that the crossovers associated with the low order Laurent coefficients are shared by many quantities, but that the

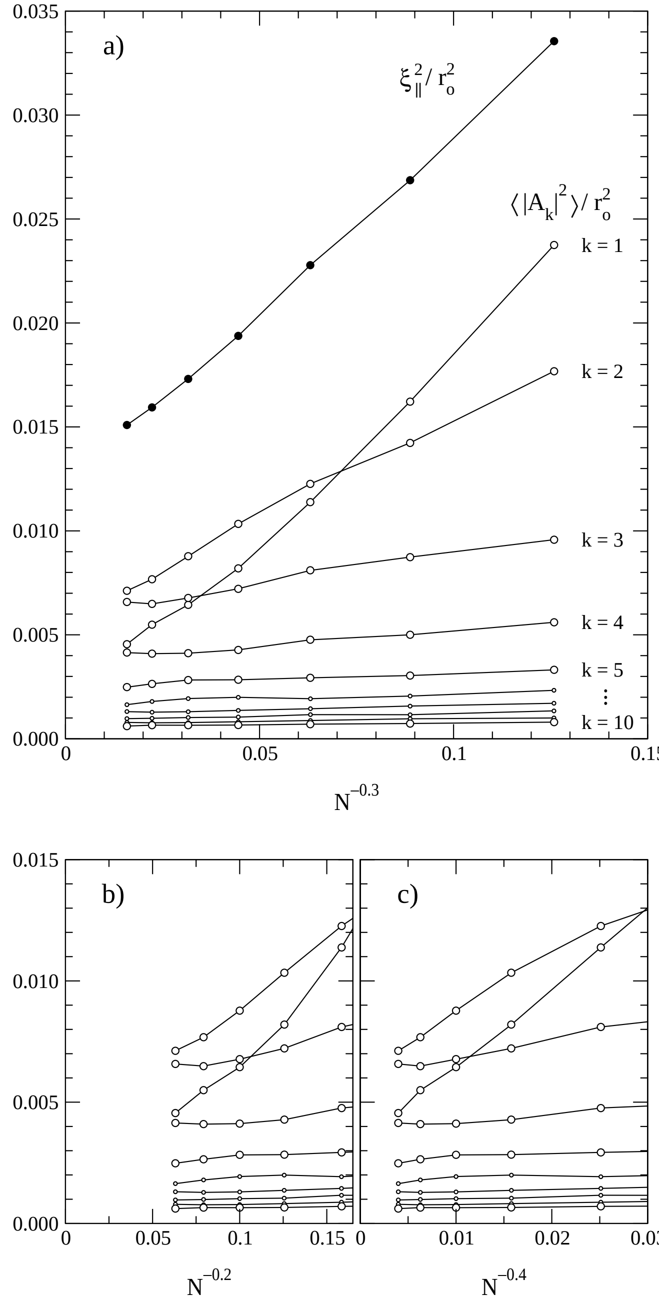

FIG. 2. (a) Crossover of the Laurent coefficients (open circles) and $\xi_\parallel^2$ (filled circles). $\xi_\parallel^2$ is approximated by the first ten terms of Eq. (4). The clusters range from $N = 10^3 - 10^6$ in steps of $\sqrt{10}$; and for each size an ensemble of 1000 was used. (b),(c) Laurent coefficients against $N^{-0.2}, N^{-0.4}$. All of the plots become straight within the range 0.3 ± 0.1 for the exponent ϕ.

asymptotic behavior is that all the characteristic length scales of the growth will scale as $N^{1/D}$.

In fact, this behavior is even more general than it appears. There is another characteristic quantity with units of squared length, namely, the ensemble fluctuation of the squared radius [4]:

$$\delta(r_0^2) \equiv [\langle r_0^4 \rangle - \langle r_0^2 \rangle^2]^{1/2}, \quad (9)$$

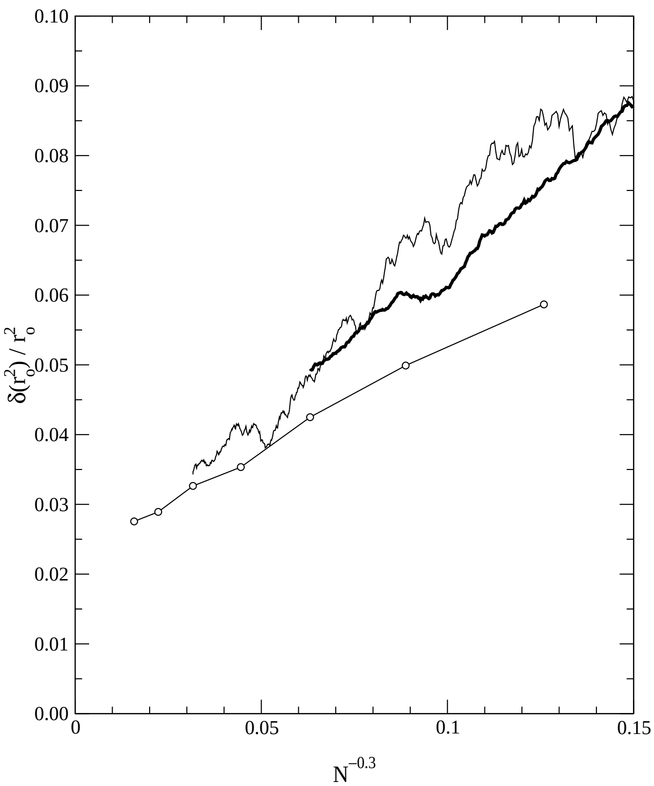

FIG. 3. Relative distribution width of the radius: $\delta(r_0^2)/r_0^2$. The distribution width $\delta(r_0^2)$ has the same correction to scaling as other squared lengths; see Fig. 2. The circles are conventionally grown DLAs averaged over 1000 clusters with $N \leq 10^6$; continuous lines, the HL method, 400 clusters with $N \leq 10^4$ (thick line), and 30 clusters with $N \leq 10^5$ (thin line).

where $\langle\,\rangle$ denotes an ensemble average. The significance of $\delta(r_0^2)$ is that it is the spread, for different clusters in the ensemble, of $r_0^2 N^{-2/D}$. This is not the same sort of object as those that we have been discussing, which are averaged properties of individual clusters. As was discussed in Ref. [4], $\delta(r_0^2)$ appears to scale more slowly than r_0^2. Note, however, that Ref. [4] used the HL method, so that only small N were available.

If we go to large N we find that $\delta(r_0^2)$ acts in the same way as $\xi_\parallel^2$, with the same correction to scaling: The apparent ensemble sharpening of the radius is also a crossover. This is shown in Fig. 3. Once more, directly generating the charge distribution with random walkers allowed us to go to large N and reveal the crossover, which was not evident in the HL method.

These results, and the last one in particular, give rise to the suspicion that all of the slowly scaling quantities in DLA growth are slaves to some underlying variable. Such a view was proposed some time ago in terms of a renormalization group by Barker and Ball [14]. In a future publication we will elaborate on this idea [15]. We think that the considerations in this paper can be extended to give rise to a very detailed understanding of DLA clusters.

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