Pattern formation is a common feature of diverse systems driven far from equilibrium.

Rayleigh-Bénard convection has served as a useful canonical example of pattern formation.

In this lecture I plan to give a basic introduction to pattern formation by addressing the question: “What happens to a macroscopic system as we drive it further away from equilibrium?”
Today’s concepts:
- patterns
- equilibrium v. nonequilibrium
- linear instability
- nonlinear saturation
- stability balloons
- pattern competition

Stationary Ideal Patterns

Stripes       Hexagons       Squares

cf. Bajaj et al. Ahlers website Ahlers website
Transients and disordered patterns

Onset of chaos in small systems

\[ R = 2804 \quad R = 6949 \]
Spatiotemporal chaos

Spiral Defect Chaos    Domain Chaos

Equilibrium, Nonequilibrium, and Far From Equilibrium
Open systems characterized by:

- Energy input, interconversion, and output
- Transport of matter and energy

Pattern formation is a common feature of systems far from equilibrium.

Equilibrium

- macroscopically uniform
- no fluxes of matter or energy
- dissipative processes return weakly perturbed system towards steady state (Onsager theory)
  - Thermal conductivity $\rightarrow$ Uniform temperature
  - Viscosity $\rightarrow$ zero velocity
  - Diffusion $\rightarrow$ uniform species concentration

Typically get exponential decay to equilibrium

$$ u(x, t) = \sum_n u_n e^{i\mathbf{q}_n \cdot \mathbf{x}} e^{-\kappa \mathbf{q}_n^2 t} $$

Far from Equilibrium

- Exponential growth of disturbances
Pattern formation occurs when the growing perturbation about the spatially uniform state has spatial structure (a mode with nonzero wave vector).
Dynamical Equations

I shall confine my discussion to systems far from equilibrium that are macroscopic and continuous.

These are defined by dynamical equations that

- Reflect the laws of thermodynamics and the return to (local) equilibrium
- Are the familiar phenomenological equations

Leads us to the study of nonlinear, determinisitic, PDEs

Equations for Convection (Boussinesq)

\[ \sigma^{-1} \left( \partial_t + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\nabla p + RT \mathbf{z} + \nabla^2 \mathbf{v} \]

\[ (\partial_t + \mathbf{v} \cdot \nabla) T = \nabla^2 T \]

\[ \nabla \cdot \mathbf{v} = 0 \]

Boundary conditions

\[ \mathbf{v} = 0 \text{ at } z = 0, 1 \]

\[ T = \begin{cases} 
1 & \text{at } z = 0 \\
0 & \text{at } z = 1 
\end{cases} \]

Conducting solution: \( \mathbf{v} = 0, T = 1 - z \)
A first approach to patterns: linear stability analysis

1. Find equations of motion of the physical variables $u(x, y, z, t)$

2. Find the uniform base solution $u_b(z)$ independent of $x, y, t$

3. Focus on deviation from $u_b$

$$u(x, t) = u_b(z) + \delta u(x, t)$$

4. Linearize equations about $u_b$, i.e. substitute into equations of part (1) and keep all terms with just one power of $\delta u$. This will give an equation of the form

$$\partial_t \delta u = \mathbf{L} \delta u$$

where $\mathbf{L}$ may involve $u_b$ and include spatial derivatives acting on $\delta u$

5. Since $\mathbf{L}$ is independent of $x, y, t$ we can find solutions

$$\delta u_q(x_\perp, z, t) = u_q(z) e^{iqx_\perp} e^{i\sigma qt}$$

$\text{Re } \sigma_q$ gives exponential growth or decay

$\text{Im } \sigma_q = -\omega_q$ gives oscillations, waves $e^{i(qx_\perp - \omega qt)}$

$\text{Im } \sigma_q = 0 \implies$ Stationary instability

$\text{Im } \sigma_q \neq 0 \implies$ Oscillatory instability

For this lecture I will look at the case of stationary instability
Exponential growth: \[ \exp[\sigma_q t] \]

\[ \lambda = \frac{2\pi}{q} \]

Rayleigh’s Calculation

\[ \delta T_q(x, z) = \left( q^2 + \pi^2 \right)^2 \cos(\pi z) \cos(q x), \]
\[ \delta u_q(x, z) = q^2 \cos(\pi z) \cos(q x), \]
\[ \delta u_q(x, z) = -i \pi q \sin(\pi z) \sin(q x). \]
\[(\sigma^{-1}\sigma_q + \pi^2 + q^2)(\sigma_q + \pi^2 + q^2) - Rq^2/(\pi^2 + q^2) = 0\]

For \(R\) near \(R_c\) and \(q\) near \(q_c\)

\[
\text{Re } \sigma_q = \tau_0^{-1}[\varepsilon - \xi_0^2 (q - q_c)^2] \quad \text{with} \quad \varepsilon = \frac{R - R_c}{R_c}
\]
For $R$ near $R_c$ and $q$ near $q_c$,

$$\text{Re } \sigma_q = \tau_0^{-1} [\varepsilon - \xi_0^2 (q - q_c)^2]$$

with

$$\varepsilon = \frac{R - R_c}{R_c}$$

Setting $\text{Re } \sigma_q = 0$ defines the neutral stability curve $R = R_c(q)$

$$R_c(q) = \frac{(q^2 + \pi^2)^3}{q^2} \quad \Rightarrow \quad R_c = \frac{27\pi^4}{4}, \quad q_c = \frac{\pi}{\sqrt{2}}$$
Linear stability theory is often a useful first step in understanding pattern formation:

- Often is quite easy to do either analytically or numerically
- Displays the important physical processes
- Gives the length scale of the pattern formation $1/q_c$

But:

- Leaves us with unphysical exponentially growing solutions
- Not all pattern formation phenomena can be connected back to the linear onset

Nonlinearity
\[ \text{Re } \sigma_q > 0 \]

\[ \text{Re } \sigma_q < 0 \]

\[ q \]

\[ R \]

\[ R_c \]

\[ q_c \]
Benasque PHYSBIO 2003: Pattern Formation in Rayleigh-Bénard Convection - Lecture 1

Diagram showing the behavior of $\text{Re } \sigma_q$ with $q$ for different values of $R_c$ and $q_N$.
Patterns exist. Are they stable?

No patterns

nonlinear states
The diagram illustrates the stability transitions in the context of Rayleigh-Bénard convection. It shows a graph with axes $R$ and $q_c$, where $R_c$ denotes a critical value. The graph includes regions labeled as $E=Eckhaus$, $E=ZigZag$, and indicates stable and unstable states at different $q_c$ values. The $E=Eckhaus$ region transitions into $E=ZigZag$, with $q_c$ values marking the boundaries.
The diagram shows a phase diagram with various regions labeled:

- **E**: Eckhaus
- **Z**: ZigZag
- **SV**: Skew Varicose
- **O**: Oscillatory

The diagram includes a shaded region labeled as the stable band, which transitions through different states as indicated by the symbols and arrows.

The axes are labeled as follows:
- **R**
- **q**
- **R_c**

The states and transitions are marked with symbols such as **E**, **Z**, **SV**, and **O**.
Summary so far

- Uniform state unstable to growth of perturbation with wave vector $\mathbf{q}$ for $R > R_c(\mathbf{q})$
- Neutral stability curve $\mathbf{q}_{N \pm}(R)$ defined by $R_c(\mathbf{q}_{N \pm}) = R$
- Saturated, nonlinear states exist for $\mathbf{q}_{N -}(R) < q < \mathbf{q}_{N +}(R)$
- Nonlinear states are stable in a restricted band of wave vectors $\mathbf{q}_{S -}(R) < q < \mathbf{q}_{S +}(R)$ (the stability balloon)
- Near onset the instabilities take on a universal form (Eckhaus and ZigZag)

$$q_{E +} - q_c = q_{E -} = \frac{1}{\sqrt{3}}(q_{N +} - q_c) \propto \varepsilon^{1/2}$$

$$q_{Z} - q_c = 0 \times \varepsilon^{1/2} + \ldots \times \varepsilon$$

- Away from onset the specifics of the system will be important

Role of Symmetry

Full rotational symmetry in plane

(a) R

(b) $q_y$

No patterns

Patterns

$q_c$

$q$

$q_x$

$q_y$
At the linear level any superposition of modes with $|\mathbf{q}| \simeq q_c$ gives a (time dependent) solution.

Nonlinearity will:

- Saturate growth
- Lead to mode competition

How to proceed?

- There is in general no way to list all possible solutions of a nonlinear PDE
- Use symmetry to suggest possible structures
- Test stability
It should be noted that many of the same questions come up in equilibrium physics in discussing crystal structure. Similarities:

1. Propose structure, and test for stability
2. Similar structures
3. Group theory is a useful tool

Differences in nonequilibrium systems:

1. No free energy to compare states, and choose between multistable states
2. Transitions often continuous (second order) or weakly first order, so that analysis in terms of wave vectors at $q_e$ makes sense
3. Often interested in two dimensional structures
Anisotropy direction (only $x \rightarrow -x$ symmetry)

Conclusions

Introduced a class of pattern formation in systems that are far from equilibrium globally, but are near equilibrium locally, so that they are described by familiar continuum equations.

Study of behavior of nonlinear PDEs.

Asking what happens as we drive a system away from the thermodynamic equilibrium state leads to an understanding of pattern formation in terms of linear instability from a uniform state.

Developed a qualitative picture in terms of existence and stability of simple nonlinear states with wave number $q$.

Stressed the importance of symmetry.

For systems with rotational symmetry different patterns (stripes, lattices, quasilattices) must be considered.

Many nonequilibrium patterns cannot be continuously connected to the equilibrium state by varying a convenient parameter, but many of the concepts and qualitative results we have developed will continue to apply.