Pattern Formation in Rayleigh-Bénard Convection

Lecture 3

Chaos
Outline

- Low dimensional chaos
- Pattern Chaos
- Spatiotemporal chaos
Outline

• Low dimensional chaos

• Pattern Chaos

• Spatiotemporal chaos

Collaborators: Marc Bourzutschky, Keng-Hwee Chiam, Henry Greenside, Roman Grigoriev, Pierre Hohenberg, Michael Louie, Dan Meiron, Mark Paul, Yuhai Tu.

Support: NSF and DOE
Lorenz Chaos

\[ T(x, z, t) \simeq -rz \]
Lorenz Chaos

\[ T(x, z, t) \simeq -r z + 9\pi^3 \sqrt{3} Y(t) \cos(\pi z) \cos\left(\frac{\pi}{\sqrt{2}} x\right) \]
Lorenz Chaos

\[ T(x, z, t) \simeq -rz + 9\pi^3 \sqrt{3} Y(t) \cos(\pi z) \cos\left(\frac{\pi}{\sqrt{2}} x\right) + \frac{27\pi^3}{4} Z(t) \sin(2\pi z) \]
Lorenz Chaos

\[ T(x, z, t) \simeq -rz + 9\pi^3\sqrt{3}Y(t)\cos(\pi z)\cos\left(\frac{\pi}{\sqrt{2}}x\right) + \frac{27\pi^3}{4}Z(t)\sin(2\pi z) \]

\[ \psi(x, z, t) = 2\sqrt{6}X(t)\cos(\pi z)\sin\left(\frac{\pi}{\sqrt{2}}x\right) \]
Lorenz Chaos

\[
T(x, z, t) \simeq -rz + 9\pi^3 \sqrt{3} Y(t) \cos(\pi z) \cos\left(\frac{\pi}{\sqrt{2}}x\right) + \frac{27\pi^3}{4} Z(t) \sin(2\pi z)
\]

\[
\psi(x, z, t) = 2\sqrt{6} X(t) \cos(\pi z) \sin\left(\frac{\pi}{\sqrt{2}}x\right)
\]

\[
(u = -\partial\psi/\partial z, \quad v = \partial\psi/\partial x, \quad r = R/R_c)
\]
Lorenz Model

\[
\begin{align*}
\dot{X} &= -\sigma (X - Y) \\
\dot{Y} &= r X - Y - XZ \\
\dot{Z} &= b (XY - Z)
\end{align*}
\]

\(b = 8/3\) and \(\sigma\) is the Prandtl number.

“Classic” values are \(\sigma = 10\) and \(r = 27\).
The Butterfly Effect

The “sensitive dependence on initial conditions” found by Lorenz is often called the “butterfly effect”.

In fact Lorenz first said (Transactions of the New York Academy of Sciences, 1963)

One meteorologist remarked that if the theory were correct, one flap of the sea gull’s wings would be enough to alter the course of the weather forever.

By the time of Lorenz’s talk at the December 1972 meeting of the American Association for the Advancement of Science in Washington, D.C. the sea gull had evolved into the more poetic butterfly - the title of his talk was

Predictability: Does the Flap of a Butterfly’s Wings in Brazil set off a Tornado in Texas?
The Lorenz model does not describe Rayleigh-Bénard convection!

Also: Rititake homopolar dynamo
Small system chaos
Theoretical Context

**Landau (1944)**  Turbulence develops by infinite sequence of transitions adding additional temporal modes and spatial complexity

**Ruelle and Takens (1971)**  Suggested the onset of aperiodic dynamics from a low dimensional torus (quasiperiodic motion with a small number $N$ frequencies)

**Feigenbaum (1978)**  Quantitative universality for period doubling route to chaos
Some Experimental Highlights

Ahlers (1974) Transition from time independent flow to aperiodic flow at $R/R_c \sim 2$ (aspect ratio 5)

Gollub and Swinney (1975) Onset of aperiodic flow from time-periodic flow in Taylor-Couette

Maurer and Libchaber, Ahlers and Behringer (1978) Transition from quasiperiodic flow to aperiodic flow in small aspect ratio convection

Lichaber, Laroche, and Fauve (1982) Quantitative demonstration of the Fiegenbaum period doubling route to chaos
Larger System Chaos

In “large” systems presumably the chaos becomes higher dimensional.

We might divide up the phenomena into two classes:

**Pattern chaos**  global properties of spatial patterns are important, perhaps induced by boundary effects

**Spatiotemporal chaos**  statistical aspects of local units are more important \((\Gamma \to \infty)\)

In either case we might be interested in

- the mechanism of the dynamics
- characterizing the dynamics
Pattern chaos: convection in intermediate aspect ratio cylinders

• First experiments: $\Gamma = 5.27$ cell, cryogenic (normal) liquid $He^4$ as fluid. High precision heat flow measurements (no flow visualization).

• Onset of aperiodic time dependence in low Reynolds number flow: relevance of chaos to “real” (continuum) systems.

• Power law decrease of power spectrum $P(f) \sim f^{-4}$

• Aspect ratio dependence of the onset of time dependence

<table>
<thead>
<tr>
<th>$\Gamma$</th>
<th>2</th>
<th>5</th>
<th>57</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_t$</td>
<td>$10R_c$</td>
<td>$2R_c$</td>
<td>$1.1R_c$</td>
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</table>

• Flow visualization (Argon, $\Gamma = 7.66$)
(from Ahlers 1974)
(from Ahlers and Behringer 1978)
Gao and Behringer (1984)
(from Ahlers and Behringer 1978)
Numerical Simulations

(Paul, MCC, Fischer and Greenside)

- $\Gamma = 4.72, \sigma = 0.78, 2600 \lesssim R \lesssim 7000$
- Conducting sidewalls
- Random thermal perturbation initial conditions
- Simulation time $\sim 100\tau_h$
  - Simulation time $\sim 12$ hours on 32 processors
  - Experiment time $\sim 172$ hours or $\sim 1$ week
\[ \Gamma = 4.72 \]
\[ \sigma = 0.78 \text{ (Helium)} \]
Random Initial Conditions

\[ \begin{array}{c}
R = 6949 \\
R = 4343 \\
R = 3474 \\
R = 3127 \\
R = 2804 \\
R = 2606 \\
\end{array} \]
\[ R = 3127(?) \quad \quad \quad \quad \quad \quad R = 6949 \]
Power Spectrum

- Simulations of low dimensional chaos (e.g. Lorenz model) show exponential decaying power spectrum.
- Power law power spectrum easily obtained from stochastic models (white-noise driven oscillator, etc.)

<table>
<thead>
<tr>
<th>Deterministic Chaos</th>
<th>? ⇒?</th>
<th>Exponential decay</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stochastic Noise</td>
<td>? ⇒?</td>
<td>Power law decay</td>
</tr>
</tbody>
</table>
Simulation Results

Simulations reproduce experimental results....
Simulation yields a power law over the range accessible to experiment

\[ \log_{10} P(\omega) \]

\[ \frac{R}{R_c} = 4.62 \quad \Gamma = 4.72 \]

\[ P(\omega) \sim \omega^{-4} \]

\[ R = 6949 \]

Conducting sidewalls

\[ \Gamma = 4.72, \sigma = 0.78, R = 6949 \]

\[ R/R_c = 4.0 \]
When larger frequencies are included an exponential tail is found

\[ \Gamma = 4.72, \sigma = 0.78, R = 6949 \]

Conducting sidewalls

\[ R/R_c = 4.0 \]
Exponential tail not seen in experiment because of instrumental noise floor
Where does the power law come from?

Power law arises from quasi-discontinuous changes in the slope of $N(t)$ on a $t = 0.1 - 1$ time scale associated with roll pinch-off events.

This is clearest to see for the low Rayleigh number where the motion is periodic, but again the power spectrum has a power law fall off:

- Spectrogram
- Details of time series $N(t)$
- Windowed power spectrum
- Compare periodic and chaotic events
Periodic dynamics $R = 2804$
Power law spectra

\[ P(\nu) \propto \nu^{1/2} \]

\[ \Gamma = 4.72, R = 2804, \sigma = 0.78 \]

- Conducting sidewalls
- No overlapping
- Linear detrending
Spectrogram
Time series $N(t)$

$$R = 2804, \Gamma = 4.72, \sigma = 0.78$$

Conducting

- **a)** dislocation nucleation
- **b)** dislocations climb
- **c)** both dislocations are at the lateral walls
- **d)** first annihilation
- **e)** the other dislocation slowly glides into the same wall foci
- **f)** process repeats

$\text{second annihilation}$

Note: Dislocation glide alternates left and right, the portion highlighted here goes right.
Windowed power spectra

\[ \langle P(\nu) \rangle \]

- \( t = 594 \), Nucleation
- \( t = 639 \), Annihilation
- \( t = 700 \), Glide
- \( \nu^{-4} \)
- \( \langle P(\nu) \rangle \)
Compare periodic and chaotic events

\[ R = 2804, \sigma = 0.78, \Gamma = 4.72 \]
Conducting sidewalls

\[ R = 6949, \sigma = 0.78, \Gamma = 4.72 \]
Conducting sidewalls
Mechanism of Dynamics

- Pan-Am texture with roll pinch-off events creating dislocation pairs (Flow visualization, Pocheau, Le Gal, and Croquette 1985)

- Mean flow compresses rolls outside of stable band (Model equations, Greenside, MCC, Coughran 1985)

- Theoretical analysis of mean flow (Pocheau and Davidaud 1997)

- Numerical simulation of importance of mean flow (Paul, MCC, Fischer, and Greenside 2001)
Generalized Swift-Hohenberg Simulations
Convection Simulations

3 convection cells with different side wall conditions: (a) rigid; (b) finned; and (c) ramped. Case (a) is dynamic, the others static.
Spatiotemporal chaos
Spatiotemporal chaos

Might be defined as the dynamics, disordered in time and space, of a large aspect ratio system (i.e. one that is large compared to the size of a basic chaotic element)

Natural examples

- The atmosphere and ocean (weather, climate etc.)
- Heart fibrillation

Spatiotemporal chaos is a new paradigm of unpredictable dynamics. (What effect does the butterfly really have?)
Spatiotemporal chaos and turbulence both correspond to $L \rightarrow \infty$

Define length scales:

- Energy injection scale $L_I$
- Energy dissipation scale $L_D$
- System size $L$

Spatiotemporal chaos $L >> L_D \sim L_I$

Turbulence $L \sim L_I >> L_D$

“Spatiotemporal turbulence” $L >> L_I >> L_D$
Spatiotemporal Chaos v. Turbulence

Energy Injection

Energy Dissipation

L_D

Turbulence

Spatiotemporal Chaos

L_D

Spatiotemporal Turbulence
Systems

- Coupled Maps
  \[x_{i}^{(n+1)} = f(x_{i}^{(n)}) + g \times (x_{i-1}^{(n)} - 2x_{i}^{(n)} + x_{i+1}^{(n)})\]

- PDE simulations
  - Complex Ginzburg-Landau Equation
    \[\partial_t A = A + (1 + ic_1)\nabla^2 A - (1 - ic_3)|A|^2 A\]
  - Kuramoto-Sivashinsky equation
    \[\partial_t u = -\partial_x^2 u - \partial_x^4 u - u \partial_x u\]

- Physical systems (experiment and numerics)
Examples from convection

- Spiral Defect Chaos (experiment, simulations)
- Domain Chaos (model simulations: stripes, orientations, walls; convection simulations)
Issues

• System-specific questions

• Definition and characterization
  – How to narrow the phenomena
  – How to decide if theory, simulation, and experiment match

• Transitions to spatiotemporal chaos and between different chaotic states

• Thermodynamics and hydrodynamics (cf. Granular Flows)

• Control
Issues

• System-specific questions

• Definition and characterization
  – How to narrow the phenomena
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• Transitions to spatiotemporal chaos and between different chaotic states

• Thermodynamics and hydrodynamics (cf. Granular Flows)

• Control

Ideas and methods from dynamical systems, statistical mechanics, phase transition theory …
Spatiotemporal Chaos in Rotating Rayleigh-Bénard Convection
Rayleigh Number

stripes (convection)

$R_c$
The graph illustrates the relationship between the Rayleigh Number and Rotation Rate in the context of Rayleigh-Bénard convection. The horizontal axis represents the Rotation Rate, while the vertical axis represents the Rayleigh Number.

- Below the critical rotation rate $R_c$, there is no pattern (conduction).
- Above $R_c$, the pattern changes from no pattern to stripes (convection).

The curve on the graph shows how the Rayleigh Number increases with the Rotation Rate, indicating the transition from conduction to convection.
Rotation Rate

Rayleigh Number

no pattern (conduction)

KL

stripes (convection)

R_c

Rotation Rate

Rayleigh Number

- no pattern (conduction)
- stripes (convection)
- domain chaos

KL

R_c

Back
Forward
Rayleigh Number

Rotation Rate

- no pattern (conduction)
- stripes (convection)
- domain chaos
- spiral chaos
- KL

$R_C$
Rotation Rate

Rayleigh Number

KL

spiral
chaos

domain
chaos

stripes
(convection)

no pattern
(conduction)

$R_c$

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Rotation Rate

Rayleigh Number

no pattern (conduction)

stripes (convection)

spiral chaos
domain chaos

KL

RC

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Rotation Rate

Rayleigh Number

no pattern (conduction)

domain chaos

stripes (convection)

spiral chaos

KL

\( R_c \)
Rotation Rate

Rayleigh Number

- no pattern (conduction)
- stripes (convection)
- spiral chaos
- domain chaos
- locked domain chaos (?)

KL
Rotation Rate

Rayleigh Number

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spiral chaos

locked domain chaos (?)

domain chaos

no pattern (conduction)

stripes (convection)

no pattern (conduction)

Rc
Rotation Rate

Rayleigh Number

- no pattern (conduction)
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- KL

$R_c$
Onset of Domain Chaos (Tu and MCC)
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Amplitudes of rolls at 3 orientations $A_i(r, t), \ i = 1 \ldots 3$
Onset of Domain Chaos (Tu and MCC)

Amplitudes of rolls at 3 orientations \( A_i(\mathbf{r}, t), i = 1 \ldots 3 \)

\[ \partial_t A_1 = \epsilon A_1 \]
Onset of Domain Chaos (Tu and MCC)

Amplitudes of rolls at 3 orientations $A_i(r, t), \ i = 1 \ldots 3$

$$\partial_t A_1 = \varepsilon A_1 + \partial_{x_1}^2 A_1$$
Onset of Domain Chaos (Tu and MCC)

Amplitudes of rolls at 3 orientations $A_i(\mathbf{r}, t)$, $i = 1 \ldots 3$

$$\partial_t A_1 = \varepsilon A_1 + \partial_{x_1}^2 A_1 - A_1 (A_1^2)$$
Onset of Domain Chaos (Tu and MCC)

Amplitudes of rolls at 3 orientations $A_i(r, t), \ i = 1 \ldots 3$

$$\partial_t A_1 = \varepsilon A_1 + \partial_{x_1}^2 A_1 - A_1 (A_1^2 + g_+ A_2^2 + g_- A_3^2)$$


**Onset of Domain Chaos** (Tu and MCC)

Amplitudes of rolls at 3 orientations $A_i(r, t), i = 1 \ldots 3$

\[
\begin{align*}
\partial_t A_1 &= \varepsilon A_1 + \partial_{x_1}^2 A_1 - A_1(A_1^2 + g_+ A_2^2 + g_- A_3^2) \\
\partial_t A_2 &= \varepsilon A_2 + \partial_{x_2}^2 A_2 - A_2(A_2^2 + g_+ A_3^2 + g_- A_1^2) \\
\partial_t A_3 &= \varepsilon A_3 + \partial_{x_3}^2 A_3 - A_3(A_3^2 + g_+ A_1^2 + g_- A_2^2)
\end{align*}
\]

where $\varepsilon = (R - R_c(\Omega))/R_c(\Omega)$
Onset of Domain Chaos (Tu and MCC)

Amplitudes of rolls at 3 orientations $A_i(r, t), i = 1 \ldots 3$

\[
\begin{align*}
\partial_t A_1 &= \varepsilon A_1 + \partial_{x_1}^2 A_1 - A_1(A_1^2 + g_+ A_2^2 + g_- A_3^2) \\
\partial_t A_2 &= \varepsilon A_2 + \partial_{x_2}^2 A_2 - A_2(A_2^2 + g_+ A_3^2 + g_- A_1^2) \\
\partial_t A_3 &= \varepsilon A_3 + \partial_{x_3}^2 A_3 - A_3(A_3^2 + g_+ A_1^2 + g_- A_2^2)
\end{align*}
\]

where $\varepsilon = (R - R_c(\Omega))/R_c(\Omega)$

Rescale space, time, and amplitudes:
Rescale $X = \varepsilon^{1/2}x$, $T = \varepsilon t$, $\tilde{A} = \varepsilon^{-1/2}A$

\[
\begin{align*}
\partial_T \tilde{A}_1 &= \tilde{A}_1 + \partial_{X_1}^2 \tilde{A}_1 - \tilde{A}_1(\tilde{A}_1^2 + g_+ \tilde{A}_2^2 + g_- \tilde{A}_3^2) \\
\partial_T \tilde{A}_2 &= \tilde{A}_2 + \partial_{X_2}^2 \tilde{A}_2 - \tilde{A}_2(\tilde{A}_2^2 + g_+ \tilde{A}_3^2 + g_- \tilde{A}_1^2) \\
\partial_T \tilde{A}_3 &= \tilde{A}_3 + \partial_{X_3}^2 \tilde{A}_3 - \tilde{A}_3(\tilde{A}_3^2 + g_+ \tilde{A}_1^2 + g_- \tilde{A}_2^2)
\end{align*}
\]
Rescale $X = \varepsilon^{1/2} x$, $T = \varepsilon t$, $\bar{A} = \varepsilon^{-1/2} A$

\[ \partial_T \bar{A}_1 = \bar{A}_1 + \partial_{X_1}^2 \bar{A}_1 - \bar{A}_1 (\bar{A}_1^2 + g_+ \bar{A}_2^2 + g_- \bar{A}_3^2) \]
\[ \partial_T \bar{A}_2 = \bar{A}_2 + \partial_{X_2}^2 \bar{A}_2 - \bar{A}_2 (\bar{A}_2^2 + g_+ \bar{A}_3^2 + g_- \bar{A}_1^2) \]
\[ \partial_T \bar{A}_3 = \bar{A}_3 + \partial_{X_3}^2 \bar{A}_3 - \bar{A}_3 (\bar{A}_3^2 + g_+ \bar{A}_1^2 + g_- \bar{A}_2^2) \]

Numerical simulations show chaotic dynamics

Therefore in unscaled (physical) units

- Length scale $\xi \sim \varepsilon^{-1/2}$
- Time scale $\tau \sim \varepsilon^{-1}$
Experimental results

**Effect of finite size**

Hypothesis: deviations from the predicted scaling behavior comes from the finite size of the experimental system (MCC, Louie, and Meiron 2001)

Use numerical simulations to test a finite size scaling ansatz

$$\xi_M = \xi f(\Gamma/\xi), \quad \xi = \xi_0 \varepsilon^{-1/2}$$

Generalized Swift-Hohenberg equation (real field of two spatial dimensions $\psi(x, y; t)$)

$$\frac{\partial \psi}{\partial t} = \varepsilon \psi + (\nabla^2 + 1)^2 \psi - g_1 \psi^3$$

$$+ g_2 \hat{z} \cdot \nabla \times [(\nabla \psi)^2 \nabla \psi] + g_3 \nabla \cdot [(\nabla \psi)^2 \nabla \psi]$$
Variation with $\varepsilon$

$\varepsilon = 0.1$

$\varepsilon = 0.3$
Variation with system size
Calculation of domain size
Domain size v. $\varepsilon$

![Graph showing domain size vs. $\varepsilon$ with different symbols and lines for $\Gamma=30\pi$, $\Gamma=40\pi$, $\Gamma=50\pi$, and $\Gamma=80\pi$.]
Finite size scaling

\[ \xi_M = \xi f(\Gamma/\xi), \quad \xi = \xi_0 \varepsilon^{-1/2} \]
Characterizing Spatiotemporal Chaos

**Methods from statistical physics:** Correlation lengths and times, etc.

**Methods from dynamical systems:** Lyapunov exponents and attractor dimensions.

Quantifies sensitive dependence on initial conditions.
**Lyapunov Exponent**

\[ \lambda = \lim_{t_f \to \infty} \frac{1}{t_f - t_0} \ln \left| \frac{\delta u_f}{\delta u_0} \right| \]
Lyapunov vector for spiral defect chaos

(from Keng-Hwee Chiam, Caltech thesis 2003)
Lyapunov spectrum for spiral defect chaos

(from Egolf et al. 2000)
Dimensions

Most definitions of fractal dimension involve the geometry of the chaotic attractor in phase space, which is inaccessible in spatiotemporal chaos.

The Lyapunov dimension:

- is accessible to simulations (but probably not to experiment)
- is usually equal to the information dimension
- is expected to be extensive for large system sizes (Ruelle, 1982)
Lyapunov dimension

Define $\mu(n) = \sum_{i=1}^{n} \lambda_i$ \((\lambda_1 \geq \lambda_2 \cdots)\) with $\lambda_i$ the $i$th Lyapunov exponent.

$D_L$ is the interpolated value of $n$ giving $\mu = 0$ (the dimension of the volume that neither grows nor shrinks under the evolution).
Lyapunov dimension for spiral defect chaos

(Egolf et al. 2000)
Microextensive scaling for the 1d Kuramoto-Sivashinsky equation

(from Tajima and Greenside 2000)
Microextensive scaling

Interesting questions:

• Are there (tiny) windows of periodic orbits or chaotic orbits of non-scaling dimension so that smooth variation is only in the $L \to \infty$ limit, or is the variation smooth at finite $L$?

• Can we use this to define spatiotemporal chaos for finite $L$?

• Does spatiotemporal chaos in Rayleigh-Bénard Convection show microextensive scaling of the Lyapunov dimension?
**Conclusions**

Small system chaos

- Lorenz model
- experimental demonstration of chaos in continuum systems

Pattern chaos

- aperiodic flow due to pattern dynamics
- power law tail in spectrum due to defect nucleation events

Spatiotemporal chaos

- new paradigm for unpredictable dynamics (cf. chaos, turbulence)
- quantitative predictions for domain chaos in rotating convection, but discrepancies with experiment remain
- Lyapunov exponent, vector and dimension
References

Small system chaos

E. Lorenz  J. Atmos. Sciences, 20, 130 (1963)


Pattern Chaos


Spatiotemporal chaos

Domain Chaos


Lyapunov exponents and dimension
