Metrological entanglement criteria

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September 8, 2025

Metrological precision vs. multipartite entanglement

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"Alternatives of entanglement depth and metrological entanglement criteria"

Szalay, Tóth, Quantum **9**, 1718 (2025)

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Convex metrological entanglement criteria

- estimation of parameter θ in the dynamics $e^{-iA\theta}\rho e^{iA\theta}$
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- qantum Fisher information is the convex roof of the variance
- convex bound by the convex roof of the original bound

$$F_{\mathrm{Q}}(\rho, J^{\mathrm{z}})/n \leq D_{\mathrm{avg}}^{\mathrm{oF}}(\rho) \leq D^{\mathrm{oF}}(\rho)$$

The meaning of the quantities

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average size of entangled subsystems (ASES)

weaker bound $F_Q(\rho, J^z)/n \leq D(\rho)$

- $\rho_\epsilon := \epsilon\pi_k + (1-\epsilon)\rho_1$ for $\epsilon > 0$
  with $\pi_k := |\psi_k\rangle\langle\psi_k|$ k-producible and ρ_1 1-producible, $\mathrm{Tr}(\pi_k \rho_1) = 0$

Convex vs. original: $F_\mathsf{Q}(\rho, J^\mathsf{z})/n \leq D^{\mathsf{oF}}(\rho) \leq D(\rho)$

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- ρ_ϵ is not k'-producible for $k' < k$, so $D(\rho_\epsilon) = k$
- ρ_ϵ is much less entangled as π_k itself, a much lower F_{Q}/n is expected

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stronger bound $F_\mathrm{Q}(\rho, J^\mathrm{z})/n \le D^{\mathrm{oF}}(\rho)$

- for all pure decompositions $\rho = \sum_j p_j \pi_j$, with $q_k = \sum_{j:D(\pi_j)=k} p_j$

$$F_\mathrm{Q}(\rho, J^\mathrm{z})/n \le D^{\mathrm{oF}}(\rho) \le \sum_j p_j D(\pi_j)$$

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- at least $2q_1$ weight of 4-producible states are needed for this
  $3 \le 1q_1 + 3q_3 + 4q_4 = q_1 + 3(1 - q_1 - q_4) + 4q_4$ leads to $2q_1 \le q_4$

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- or at least the same q_1 weight of 5-producible states
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\begin{array}{ccccccc}
 & & D^{\mathrm{oF}}(\rho) & \leq & D(\rho) & & \text{(prod.)} \\
 & & \vee\!| & & \vee\!| & & \\
\frac{1}{n}F_{\mathrm{Q}}(\rho, J^{\mathrm{z}}) & \leq & D^{\mathrm{oF}}_{\mathrm{avg}}(\rho) & \leq & D_{\mathrm{avg}}(\rho) & & \text{(avg.)}
\end{array}
$$

Thank you for your attention!

"Alternatives of entanglement depth
and metrological entanglement criteria"

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