Alternatives of entanglement depth and metrological entanglement criteria

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Overview

- multipartite entanglement — classification / qualification / quantification
- spec.: partial separability — ordered / criteria / entropic measures
- spec.spec.: one-parameter partial separability — linearly ordered / criteria / depths
- convex metrological criteria (bounds)

$$
\begin{array}{ccccccc}
 & & D^{\mathrm{oF}}(\rho) & \leq & D(\rho) & & \text{(prod.)} \\
 & & \vee\!| & & \vee\!| & & \\
F_{\mathrm{Q}}(\rho, J^{\mathrm{z}})/n & \leq & D^{\mathrm{oF}}_{\mathrm{avg}}(\rho) & \leq & D_{\mathrm{avg}}(\rho) & & \text{(avg.)}
\end{array}
$$

good precision $\Rightarrow$ strong entanglement (criterion)

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Partial separability properties

ξ-separability

- separability w.r.t. a partition: $\xi = \{X_1, X_2, \ldots, X_{|\xi|}\}$
- refinement (partial order): $\upsilon \preceq \xi$ def.: $\forall Y \in \upsilon, \exists X \in \xi : Y \subseteq X$

  $n = 2$:

ξ-separability

- separability w.r.t. a partition: $\xi = \{X_1, X_2, \dots, X_{|\xi|}\}$
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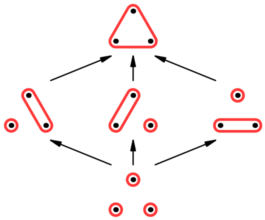
  $n = 3$:



Partial separability properties

ξ-separability

- separability w.r.t. a partition: $\xi = \{X_1, X_2, \ldots, X_{|\xi|}\}$
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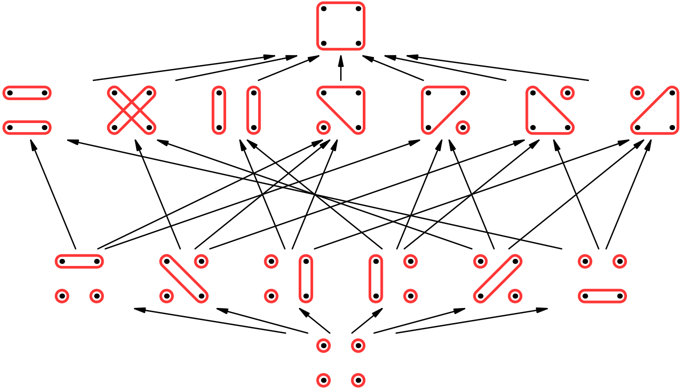
  $n = 4$:

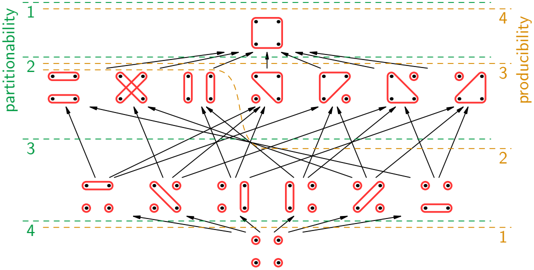


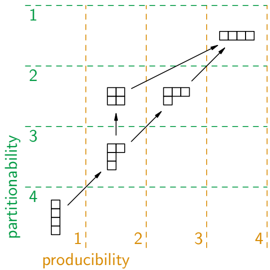
- ξ-separable states: $\mathcal{D}_\xi = \mathrm{Conv}\{\bigotimes_{X \in \xi} \rho_X\}$, LOCC-closed

$\upsilon \preceq \xi \iff \mathcal{D}_\upsilon \subseteq \mathcal{D}_\xi$

Permutation invariant properties

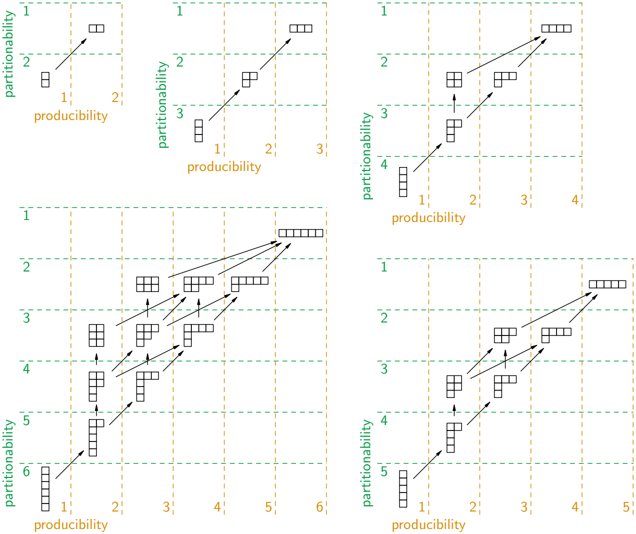
example: partitionability and producibility = minimal height and maximal width





- set partition $\xi = \{X_1, X_2, \dots\} \longmapsto$ integer partition $\hat{\xi} = \{x_1, x_2, \dots\}$ 'type of ξ' (Young diagram)
- refinement for Young diagrams, $\hat{\upsilon} \preceq \hat{\xi}$
- $\hat{\xi}$-separable states: $\mathcal{D}_{\hat{\xi}} = \operatorname{Conv} \bigcup_{\xi \leftarrow\!\shortmid \hat{\xi}} \{ \bigotimes_{X \in \xi} \rho_X \}$, LOCC-closed $\quad \hat{\upsilon} \preceq \hat{\xi} \Longleftrightarrow \mathcal{D}_{\hat{\upsilon}} \subseteq \mathcal{D}_{\hat{\xi}}$
- k-producible states: $\mathcal{D}_{k\text{-prod}} = \operatorname{Conv} \bigcup_{\xi\ k\text{-prod}} \{ \bigotimes_{X \in \xi} \rho_X \}$

Partitionability and producibility



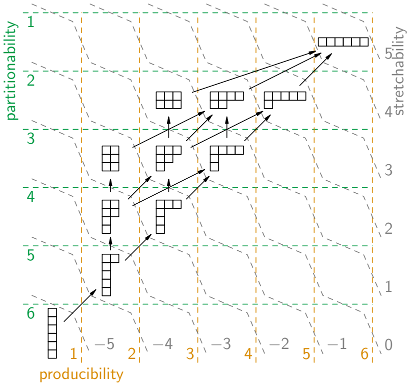
- height, width and Dyson-rank of a Young diagram

$$h(\hat{\xi}) := |\hat{\xi}| \qquad h(\hat{\upsilon}) > h(\hat{\xi})$$

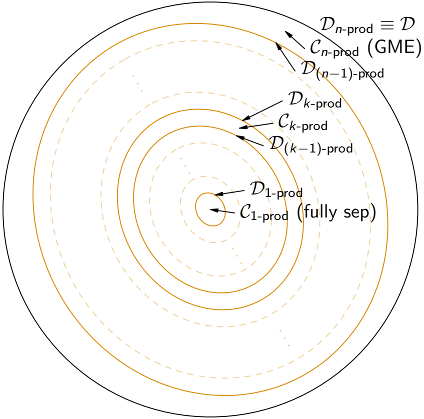
$$w(\hat{\xi}) := \max(\hat{\xi}) \qquad w(\hat{\upsilon}) \leq w(\hat{\xi})$$

$$r(\hat{\xi}) := w(\hat{\xi}) - h(\hat{\xi}) \qquad r(\hat{\upsilon}) < r(\hat{\xi})$$

- monotones $\qquad$ for $\hat{\upsilon} \prec \hat{\xi}$



Partitionability, producibility and stretchability



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$$r(\hat{\xi}) := w(\hat{\xi}) - h(\hat{\xi}) \qquad r(\hat{\upsilon}) < r(\hat{\xi})$$

- monotones for $\hat{\upsilon} \prec \hat{\xi}$
- k-producible states: $\mathcal{D}_{k\text{-prod}}$
  strictly k-producible states: $\mathcal{C}_{k\text{-prod}}$ (class)
- depth: k-value of the layer

depth of partitionability $D_{\text{part}}(\rho) := \max\{k \in \text{Ran}(h) \mid \rho \in \mathcal{D}_{k\text{-part}}\}$

depth of producibility $D_{\text{prod}}(\rho) := \min\{k \in \text{Ran}(w) \mid \rho \in \mathcal{D}_{k\text{-prod}}\} \equiv D(\rho)$

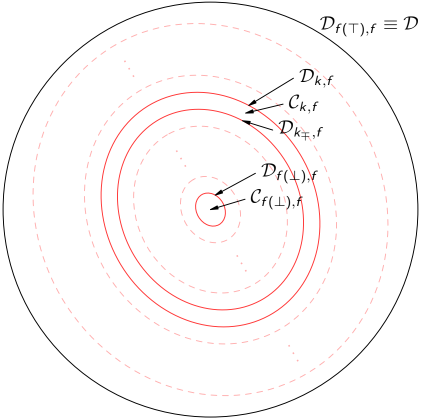
depth of stretchability $D_{\text{str}}(\rho) := \min\{k \in \text{Ran}(r) \mid \rho \in \mathcal{D}_{k\text{-str}}\}$

One-parameter entanglement properties, squareability

- generator function: f monotone

$$\hat{\upsilon} \preceq \hat{\xi} \quad \Longrightarrow \quad f(\hat{\upsilon}) \lesseqgtr f(\hat{\xi})$$

- state spaces: $\mathcal{D}_{k,f}$ (nested, LOCC closed)
- classes: $\mathcal{C}_{k,f}$ (disjoint, LOCC convertibility)
- f-entanglement depth: D_f (LOCC monotone)
  k-value of the layer



- further example: squareability, avg

$$s_2(\hat{\xi}) = \sum_{x\in\hat{\xi}} x^2 = n \sum_{x\in\hat{\xi}} \frac{x}{n} x = n\,\mathrm{avg}(\hat{\xi})$$

average size of entangled subsystems (w.r.t. picking elementary subsystem)

Depths of Formation

- *f*-entanglement depth: D_f, discrete measure
- $\rho_\epsilon := \epsilon|\psi_{\mathrm{GHZ}}\rangle\langle\psi_{\mathrm{GHZ}}| + (1-\epsilon)|\psi_{\mathrm{sep}}\rangle\langle\psi_{\mathrm{sep}}|$ with $\langle\psi_{\mathrm{sep}}|\psi_{\mathrm{GHZ}}\rangle = 0$
  problem: entanglement depth $D(\rho_\epsilon) = n$ (maximal!) for all $\epsilon > 0$
- *f*-entanglement depth of formation: convex/concave roof extension

$$D_f^{\mathrm{oF}}(\rho) := \begin{cases} \min\limits_{\{(p_j,\pi_j)\}\vdash\rho} \sum\limits_j p_j D_f(\pi_j) & \text{if } f \text{ is increasing} \\ \max\limits_{\{(p_j,\pi_j)\}\vdash\rho} \sum\limits_j p_j D_f(\pi_j) & \text{if } f \text{ is decreasing} \end{cases}$$

- problem solved: $1 < D^{\mathrm{oF}}(\rho_\epsilon) \leq \epsilon n + (1-\epsilon)1$
- note also that

$$D_f(\rho) := \begin{cases} \min\limits_{\{(p_j,\pi_j)\}\vdash\rho} \max\limits_j D_f(\pi_j) & \text{if } f \text{ is increasing} \\ \max\limits_{\{(p_j,\pi_j)\}\vdash\rho} \min\limits_j D_f(\pi_j) & \text{if } f \text{ is decreasing} \end{cases}$$

Bounds among depths

- average size of entangled subsystem $\leq$ maximal size of entangled subsystem
- convex roof extension is monotone
- convex roof extension is always smaller than the function (lower semicontinuous)

$$
\begin{array}{ccccc}
D^{\mathrm{oF}}(\rho) & \leq & D(\rho) & \text{(prod.)} \\
\rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} & \\
D^{\mathrm{oF}}_{\mathrm{avg}}(\rho) & \leq & D_{\mathrm{avg}}(\rho) & \text{(avg.)}
\end{array}
$$

Convex metrological entanglement criteria

- estimation of parameter θ in the dynamics $\rho \mapsto e^{-iA\theta}\rho e^{iA\theta}$
- Cramér-Rao bound $(\Delta\theta)^2 \geq \frac{1}{NF_{\mathrm{Q}}(\rho,A)}$, on the precision of parameter estimation by quantum Fisher information $F_{\mathrm{Q}}(\rho, A)$
- bound for collective 1/2-spin-z observable J^z for n qubits

$$F_{\mathrm{Q}}(\rho, J^z)/n \leq D(\rho)$$

- but it is actually

$$F_{\mathrm{Q}}(\rho, J^z)/n \leq D_{\mathrm{avg}}(\rho) \leq D(\rho)$$

- qantum Fisher information is the convex roof of the variance
- convex bound by the convex roof of the original bound

$$F_{\mathrm{Q}}(\rho, J^z)/n \leq D^{\mathrm{oF}}_{\mathrm{avg}}(\rho) \leq D^{\mathrm{oF}}(\rho)$$

The meaning of the quantities

$$\begin{array}{ccccccc} & & D^{\mathrm{oF}}(\rho) & \leq & D(\rho) & & \text{(prod.)} \\ & & \vee\!\mid & & \vee\!\mid & & \\ F_Q(\rho, J^z)/n & \leq & D^{\mathrm{oF}}_{\mathrm{avg}}(\rho) & \leq & D_{\mathrm{avg}}(\rho) & & \text{(avg.)} \end{array}$$

- (prod-)entanglement depth $D(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \max_j D(\pi_j)$
- (prod-)entanglement depth of formation $D^{\mathrm{oF}}(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \sum_j p_j D(\pi_j)$
- avg-entanglement depth $D_{\mathrm{avg}}(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \max_j D_{\mathrm{avg}}(\pi_j)$
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average size of entangled subsystems (ASES)

The meaning of the quantities

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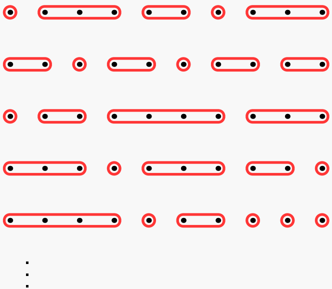
 & & \vee\! | & & \vee\! | & & \\
F_{\mathrm{Q}}(\rho, J^z)/n & \leq & D^{\mathrm{oF}}_{\mathrm{avg}}(\rho) & \leq & D_{\mathrm{avg}}(\rho) & & \text{(avg.)}
\end{array}
$$

examples

$D_{\mathrm{avg}}(\pi_j)$ $\longleftarrow$ π_j $\longmapsto$ $D(\pi_j)$



$D_{\mathrm{avg}}(\pi_j)$	$D(\pi_j)$
2.4	3
2.2	2
3	4
2.4	3
2.4	4
⋮	⋮

$$D(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \max_j D(\pi_j)$$

$$D^{\mathrm{oF}}(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \sum_j p_j D(\pi_j)$$

$$D_{\mathrm{avg}}(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \max_j D_{\mathrm{avg}}(\pi_j)$$

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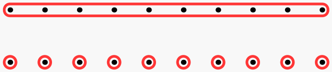
examples

$D_{\mathrm{avg}}(\pi_j)$	$\longleftarrow$	π_j	$\longmapsto$	$D(\pi_j)$
10				10
1				1

- (prod.)entanglement depth of formation $\quad D^{\mathrm{oF}}(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \sum_j p_j D(\pi_j)$
- avg-entanglement depth $\quad D_{\mathrm{avg}}(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \max_j D_{\mathrm{avg}}(\pi_j)$
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$$D(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \max_j D(\pi_j)$$

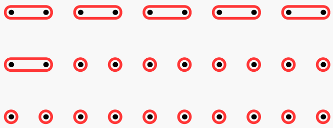
average size of entangled subsystems (ASES)

The meaning of the quantities

$$\begin{array}{ccccccc} & & D^{\mathrm{oF}}(\rho) & \leq & D(\rho) & & \text{(prod.)} \\ & & \vee\!| & & \vee\!| & & \\ F_Q(\rho, J^z)/n & \leq & D^{\mathrm{oF}}_{\mathrm{avg}}(\rho) & \leq & D_{\mathrm{avg}}(\rho) & & \text{(avg.)} \end{array}$$

examples

$D_{\mathrm{avg}}(\pi_j)$	$\longleftarrow$	π_j	$\longmapsto$	$D(\pi_j)$
2				2
1.2				2
1				1



- avg-entanglement depth of formation

$$D(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \max_j D(\pi_j)$$

$$D^{\mathrm{oF}}(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \sum_j p_j D(\pi_j)$$

$$D_{\mathrm{avg}}(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \max_j D_{\mathrm{avg}}(\pi_j)$$

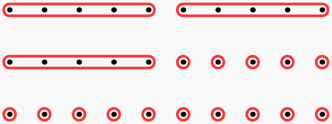
$$D^{\mathrm{oF}}_{\mathrm{avg}}(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \sum_j p_j D_{\mathrm{avg}}(\pi_j)$$

average size of entangled subsystems (ASES)

The meaning of the quantities

$$\begin{array}{ccccccc} & & D^{\mathrm{oF}}(\rho) & \le & D(\rho) & & \text{(prod.)} \\ & & \vee\!| & & \vee\!| & & \\ F_Q(\rho, J^z)/n & \le & D^{\mathrm{oF}}_{\mathrm{avg}}(\rho) & \le & D_{\mathrm{avg}}(\rho) & & \text{(avg.)} \end{array}$$

examples



$D_{\mathrm{avg}}(\pi_j)$	$\longleftarrow$ π_j $\longmapsto$	$D(\pi_j)$
5		5
3		5
1		1

- avg-entanglement depth
- avg-entanglement depth of formation

$$D(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \max_j D(\pi_j)$$

$$D^{\mathrm{oF}}(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \sum_j p_j D(\pi_j)$$

$$D_{\mathrm{avg}}(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \max_j D_{\mathrm{avg}}(\pi_j)$$

$$D^{\mathrm{oF}}_{\mathrm{avg}}(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \sum_j p_j D_{\mathrm{avg}}(\pi_j)$$

average size of entangled subsystems (ASES)

weaker bound $F_Q(\rho, J^z)/n \leq D(\rho)$

- $\rho_\epsilon := \epsilon\rho_k + (1-\epsilon)\rho_1$ for $\epsilon > 0$ with $\mathrm{Tr}(\rho_k\rho_1) = 0$
  where $D(\rho_k) = k$ (strictly k-producible) and $D(\rho_1) = 1$ (fully separable)
- ρ_ϵ is not k'-producible for $k' < k$, so $D(\rho_\epsilon) = k$ for all $\epsilon > 0$
- ρ_ϵ is much less entangled as ρ_k itself, a much lower $F_Q(\rho_\epsilon, J^z)/n$ is expected

Convex vs. original: $F_Q(\rho, J^z)/n \leq D^{oF}(\rho) \leq D(\rho)$

stronger bound $F_Q(\rho, J^z)/n \leq D^{oF}(\rho)$

- for all pure decompositions $\rho = \sum_j p_j \pi_j$, with $q_k = \sum_{j:D(\pi_j)=k} p_j$

$$F_Q(\rho, J^z)/n \leq D^{oF}(\rho) \leq \sum_j p_j D(\pi_j) = \sum_{k=1}^{n} q_k \sum_{j:D(\pi_j)=k} \frac{p_j}{q_k} D(\pi_j) = \sum_{k=1}^{n} q_k k$$

- e.g., let $n = 10$, $F_Q(\rho, J^z) \geq 30$, $F_Q(\rho, J^z)/n \geq 3$
- always exists k-producible π_j for at least one $k \geq 3$
- if there are k-producible pure states π_j for $k < 3$,
  then this has to be compensated by k-producible π_j-s for $k > 3$
- at least $2q_1$ weight of 4-producible states are needed for this
  $3 \leq 1q_1 + 3q_3 + 4q_4 = q_1 + 3(1 - q_1 - q_4) + 4q_4$ leads to $2q_1 \leq q_4$
- or at least the same q_1 weight of 5-producible states
  $3 \leq 1q_1 + 3q_3 + 5q_5 = q_1 + 3(1 - q_1 - q_5) + 5q_5$ leads to $q_1 \leq q_5$

Take home message

- k-producibility, k-average: one-parameter partial separability properties
- (there are many more, partitionability, producibility, stretchability, avg=squareability, toughness size-Rényi/Tsallis, entanglement-Dim, entanglement-DoF. . .)
- properties naturally characterized by depth and depth of formation
- metrological entanglement criteria: metrological precision (by quantum Fisher information) vs. multipartite entanglement (by entanglement depths)

$$
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 & & \vee\!| & & \vee\!| & & \\
\frac{1}{n} F_{\mathrm{Q}}(\rho, J^{\mathrm{z}}) & \leq & D^{\mathrm{oF}}_{\mathrm{avg}}(\rho) & \leq & D_{\mathrm{avg}}(\rho) & & \text{(avg.)}
\end{array}
$$

Thank you for your attention!

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