On the separability classes of noisy GHZ-W states

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Outline

Quantum information processing

Entanglement: the characteristic trait of quantum mechanics

- entangled quantum systems: nonclassical correlations

Quantum information processing

Entanglement: the characteristic trait of quantum mechanics

- entangled quantum systems: nonclassical correlations
- classically impossible tasks can be performed
- quantum computation: fast quantum algorithms, supserdense coding, quantum cryptography, quantum teleportation
- entanglement is a resource:
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- quantum computation: fast quantum algorithms, supserdense coding, quantum cryptography, quantum teleportation
- entanglement is a resource:
  entangled *pure* states are easily prepared and easy to use
- in lab: system can not be isolated form its environment
- "environmental decoherence"
- the state of the system evolves into a *mixed* one
- How many entanglement remained in the state?
- less ambitious question: Is the state still entangled?

Quantum state

Describing a quantum system

- $\mathcal{H}$ Hilbert space ($\cong (\mathbb{C}^d, \langle\cdot|\cdot\rangle)$)
- set of quantum states $\mathcal{D} = \{\varrho : \mathcal{H} \to \mathcal{H} \mid \varrho \geq 0, \mathrm{Tr}\,\varrho = 1\}$
- observables: $A : \mathcal{H} \to \mathcal{H}$ self-adjoint operators
- description of the system through expectation values: $\langle A \rangle = \mathrm{Tr}\,\varrho A$

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Pure states

- pure state: $\operatorname{rk}\varrho = 1$, then $\varrho = |\psi\rangle\langle\psi|$, $\|\psi\|^2 = 1$
  ($\varrho^2 = \varrho$ projector onto $|\psi\rangle$)
- set of pure states: projective Hilbert space ($\mathbb{C}\mathrm{P}^{d-1}$)
- $\langle A \rangle = \operatorname{Tr} \varrho A = \langle\psi|A|\psi\rangle$

Pure and mixed states

Mixed states

- mixed state: $\mathrm{rk}\,\varrho > 1$, then $\varrho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, $\|\psi_i\|^2 = 1$,
- convex sum of pure states: $p_i > 0$, $\sum_i p_i = 1$ (weighted average)

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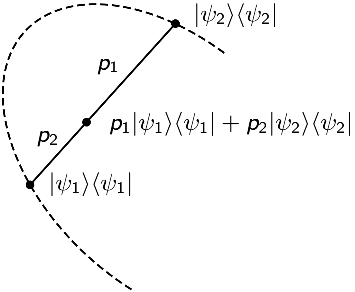

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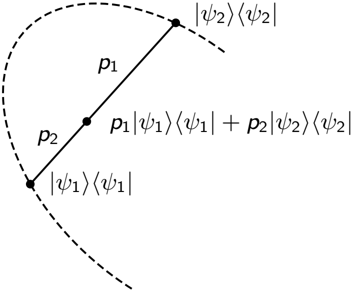

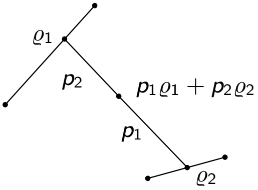

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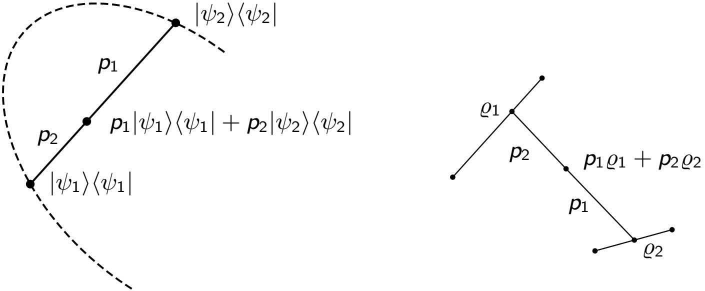

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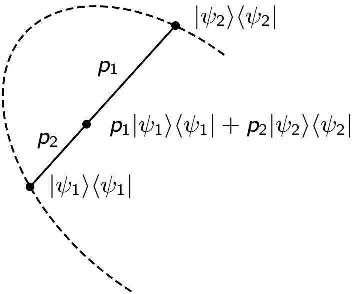
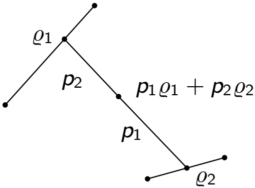
- set of all states ($\mathcal{D}$): convex hull of projective Hilbert space
- $\langle A\rangle = \mathrm{Tr}\,\varrho A = \sum_i p_i \langle\psi_i|A|\psi_i\rangle$ (system is in $|\psi_i\rangle\langle\psi_i|$ with prob. p_i)





Pure and mixed states II.

Example: one qubit

- $\mathcal{H} \cong (\mathbb{C}^2, \langle\cdot|\cdot\rangle)$
- pure states: $\mathbb{C}\mathrm{P}^1 \cong S^2$ (Bloch sphere)
- all states ($\mathcal{D}$): convex hull of Bloch sphere (Bloch ball)

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Parametrization

- $\varrho = \frac{1}{2}(\mathbb{I} + \mathbf{r}\boldsymbol{\sigma})$
- pure states: $|\mathbf{r}| = 1$
- mixed states: $|\mathbf{r}| < 1$
- center: $|\mathbf{r}| = 0$ white noise

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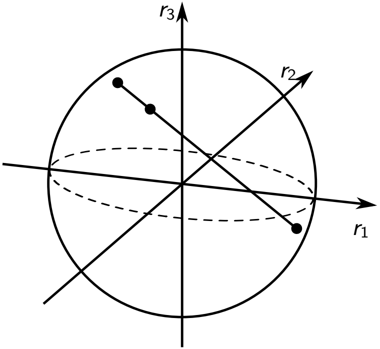

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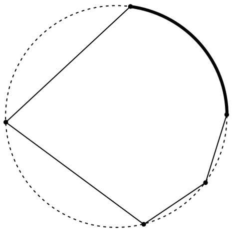

Bigger systems ($d > 2$)

- $\mathcal{H} \cong (\mathbb{C}^d, \langle\cdot|\cdot\rangle)$
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  not a sphere anymore
- but subset of zero measure *on a sphere*

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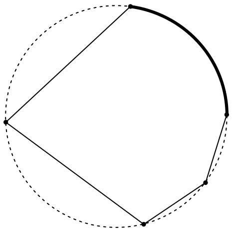

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- pure states: $\mathbb{C}\mathrm{P}^{d-1} \cong S^{2d-1}/S^1$
  not a sphere anymore
- but subset of zero measure *on a sphere*
- all states: convex hull of this
- inside: $\mathrm{rk}\,\varrho = d$
- on the boundary: $\mathrm{rk}\,\varrho < d$
  (not only pure states)
- pure states: extremal points

Composite systems

Two subsystems: $\mathcal{H} = \mathcal{H}^1 \otimes \mathcal{H}^2$

- set of quantum states $\mathcal{D} = \{\varrho : \mathcal{H} \to \mathcal{H} \mid \varrho \geq 0, \mathrm{Tr}\, \varrho = 1\}$
- $\varrho \in \mathcal{D}$ is separable iff $\varrho = \sum_i p_i \varrho_i^1 \otimes \varrho_i^2$ decomp. exist (not unique)
- pure states: $\varrho = |\psi\rangle\langle\psi|$ separable then $|\psi\rangle = |\psi^1\psi^2\rangle$
- then mixed state is separable iff $\varrho = \sum_j q_j |\psi_j^1\psi_j^2\rangle\langle\psi_j^1\psi_j^2|$
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Geometry

- all states: $\mathcal{D}$ convex hull of pure states
- extremal points: separable and entangled pure states
- separable states: $\mathcal{D}^{\text{sep}} \subset \mathcal{D}$ convex hull of pure separable states
- entangled states: the others $\mathcal{D} \setminus \mathcal{D}^{\text{sep}}$

Composite systems II.

Examples: two qubit pure states

- $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$, $|0\rangle, |1\rangle \in \mathbb{C}^2$ orthogonal normed
- entangled: Bell states $\varrho = |B_i\rangle\langle B_i|$, where

$$|B_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad |B_1\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$|B_3\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \qquad |B_2\rangle = \frac{-i}{\sqrt{2}}(|01\rangle - |10\rangle)$$

- separable: product state $\varrho = |00\rangle\langle 00|$

Composite systems III.

Examples: two qubit mixed states

- $\varrho = p|B_0\rangle\langle B_0| + (1-p)|B_3\rangle\langle B_3|$
- separable for $p=\frac{1}{2}$: $\varrho = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|$ separable

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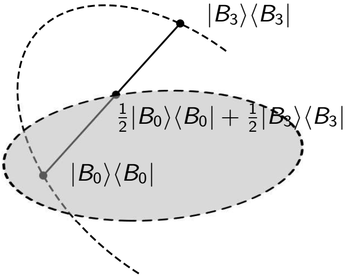

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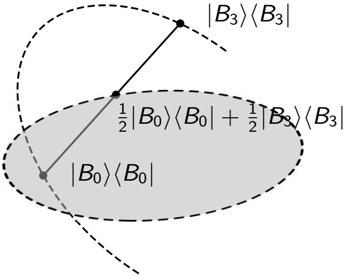
- another separable: $\varrho = \sum_i \frac{1}{4}|B_i\rangle\langle B_i| = \frac{1}{4}\mathbb{I}$



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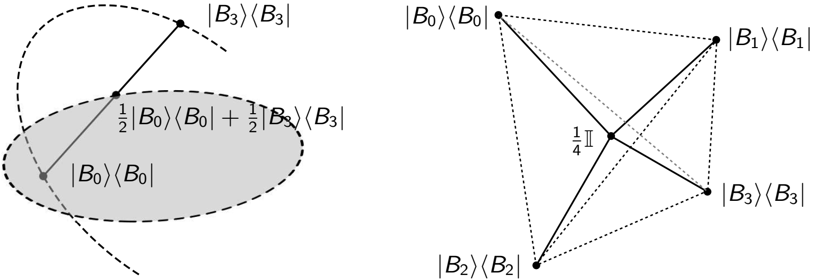
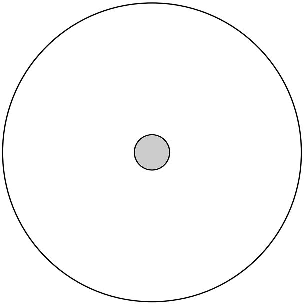


Separability of threepartite systems



- fully-separable: $\mathcal{D}^{\text{3-sep}}$
  $\varrho^{\text{3-sep}} = \sum_i p_i |\psi_i^{\text{3-sep}}\rangle\langle\psi_i^{\text{3-sep}}|$,
  $|\psi_i^{\text{3-sep}}\rangle = |\psi_i^1 \psi_i^2 \psi_i^3\rangle$

- $\mathcal{D}$-s: compact, nested convex.
- *Classes*: sets, derived from $\mathcal{D}$-s (not necessarily convex)

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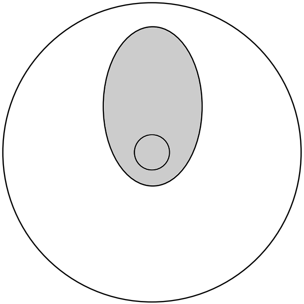

- fully-separable: $\mathcal{D}^{\text{3-sep}}$
- $1-23$-separable: $\mathcal{D}^{1-23}$,
  $\varrho^{1-23} = \sum_i p_i |\psi_i^{1-23}\rangle\langle\psi_i^{1-23}|$,
  $|\psi_i^{1-23}\rangle = |\psi_i^1 \psi_i^{23}\rangle$

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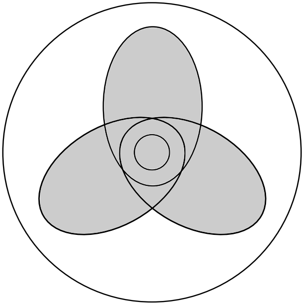

- fully-separable: $\mathcal{D}^{\text{3-sep}}$
- $1-23$-separable: $\mathcal{D}^{1-23}$,
- $\mathcal{D}^{2-13}$, $\mathcal{D}^{3-12}$

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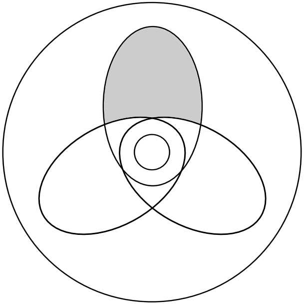

- fully-separable: $\mathcal{D}^{\text{3-sep}}$
- $1-23$-separable: $\mathcal{D}^{1-23}$,
- $\mathcal{D}^{1-23} \setminus (\mathcal{D}^{2-31} \cup \mathcal{D}^{3-12})$

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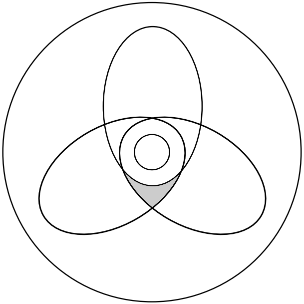

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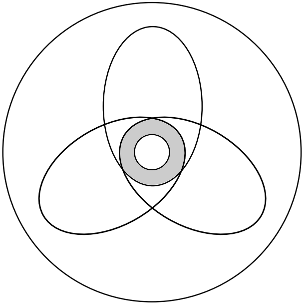

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- $(\mathcal{D}^{1-23} \cap \mathcal{D}^{2-13} \cap \mathcal{D}^{3-12}) \setminus \mathcal{D}^{\text{3-sep}}$ "semi-separable"

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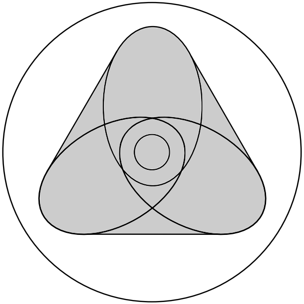

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- $(\mathcal{D}^{1-23} \cap \mathcal{D}^{2-13} \cap \mathcal{D}^{3-12}) \setminus \mathcal{D}^{\text{3-sep}}$
- biseparable: $\mathcal{D}^{\text{2-sep}} \equiv$ hull of $\mathcal{D}^{1-23} \cup \mathcal{D}^{2-13} \cup \mathcal{D}^{3-12}$
- general: k-separable: $\mathcal{D}^{k\text{-sep}}$: can be mixed using states that can be sep. *at least* k parts
- $\mathcal{D}^{\text{3-sep}} \subset \mathcal{D}^{\text{2-sep}} \subset \mathcal{D}$

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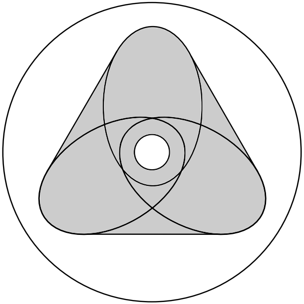

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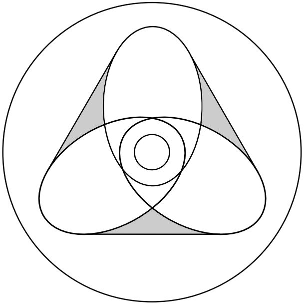

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- $\mathcal{D}^{\text{2-sep}}\setminus(\mathcal{D}^{1-23}\cup\mathcal{D}^{2-13}\cup\mathcal{D}^{3-12})$ no need of genuine three-partite entanglement

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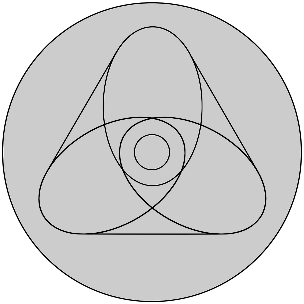

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- biseparable: $\mathcal{D}^{\text{2-sep}}$
- $\mathcal{D}^{\text{3-sep}} \subset \mathcal{D}^{\text{2-sep}} \subset \mathcal{D}$
- bisep. entangled: $\mathcal{D}^{\text{2-sep}} \setminus \mathcal{D}^{\text{3-sep}}$
- $\mathcal{D}^{\text{2-sep}} \setminus (\mathcal{D}^{1-23} \cup \mathcal{D}^{2-13} \cup \mathcal{D}^{3-12})$
- $\mathcal{D}^{\text{1-sep}} \equiv \mathcal{D}^{123} \equiv \mathcal{D}$

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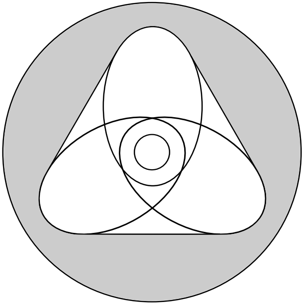

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- $\mathcal{D}^{\text{2-sep}}\setminus(\mathcal{D}^{1-23}\cup\mathcal{D}^{2-13}\cup\mathcal{D}^{3-12})$
- $\mathcal{D}^{\text{1-sep}}\equiv\mathcal{D}^{123}\equiv\mathcal{D}$
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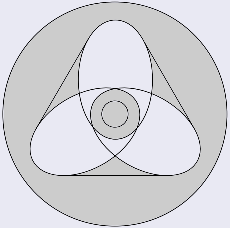
Permutation-invariant states

Simpler class structure



- some classes does not contain perm. inv. states:
- if $\varrho\in\mathcal{D}^{1-23}$ then $\varrho\in\mathcal{D}^{2-13}$ and $\varrho\in\mathcal{D}^{3-12}$ for permutation invariant ϱ
- the remaining four classes are generally nonempty

Permutation-invariant states

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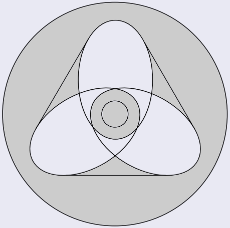
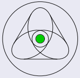
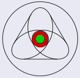
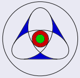


- some classes does not contain perm. inv. states:
- if $\varrho \in \mathcal{D}^{1-23}$ then $\varrho \in \mathcal{D}^{2-13}$ and $\varrho \in \mathcal{D}^{3-12}$ for permutation invariant ϱ
- the remaining four classes are generally nonempty

General scheme in the following

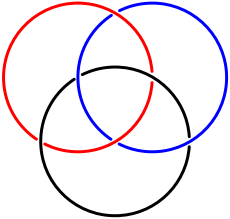
We will have criteria for three subsets:

$\mathcal{D}^{\text{3-sep}}$  $\subset$ $\mathcal{D}^{1-23} \cap \mathcal{D}^{2-13} \cap \mathcal{D}^{3-12}$  $\subset$ $\mathcal{D}^{\text{2-sep}}$ 

Important three-qubit states

GHZ-state

- Greenberger-Horne-Zeilinger
- $|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$
- maximally entangled
- separable two-partite subsystems



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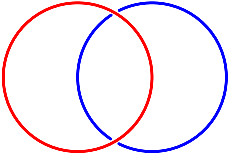

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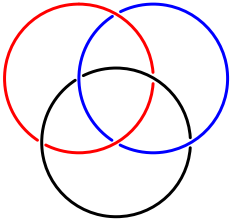

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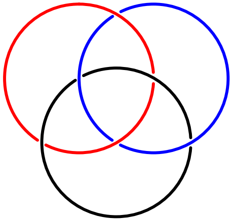

W-state

- not Werner
- $|\mathrm{W}\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$
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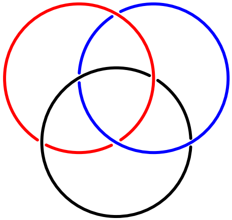

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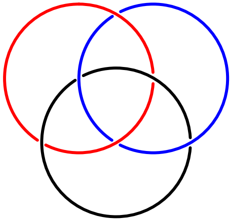

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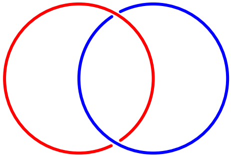

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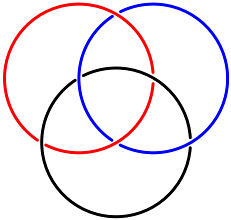

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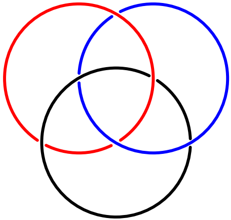

Important three-qubit states

White noise

- white noise: system is in every ortogonal state with the same prob.
- "maximally mixed"
- represented by $\frac{1}{\mathrm{Tr}\,\mathbb{I}}\mathbb{I}$
- the "center" of $\mathcal{D}$

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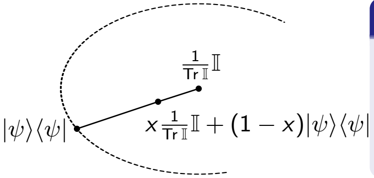


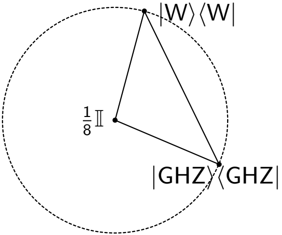
mixing with white noise

- model for environmental decoherence
- $0 \leq x \leq 1$ noise ratio
- $\varrho \mapsto \varrho_x = x\frac{1}{\mathrm{Tr}\,\mathbb{I}}\mathbb{I} + (1-x)\varrho$

Noisy GHZ-W mixture

Noisy GHZ-W mixture

- we investigate the convex hull of noise, GHZ and W states
- 2-dim. convex compact subset of $\mathcal{D}$, containing important states



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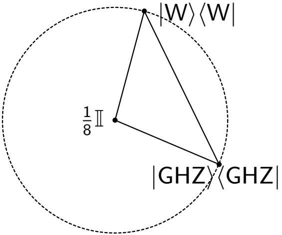
- $\varrho := d\frac{1}{8}\mathbb{I} + g|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + w|\mathrm{W}\rangle\langle\mathrm{W}|$
- $0 \leq d, g, w \leq 1$ weights, $d + g + w = 1$



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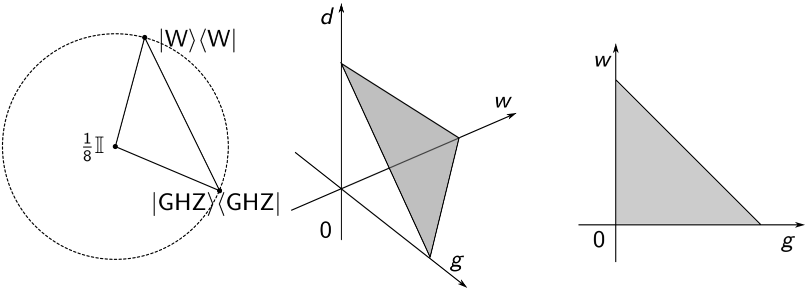

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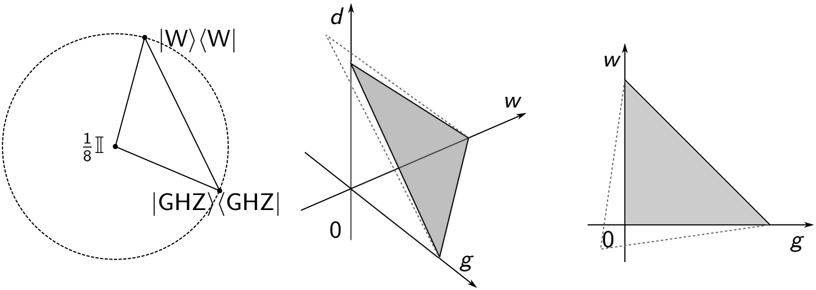

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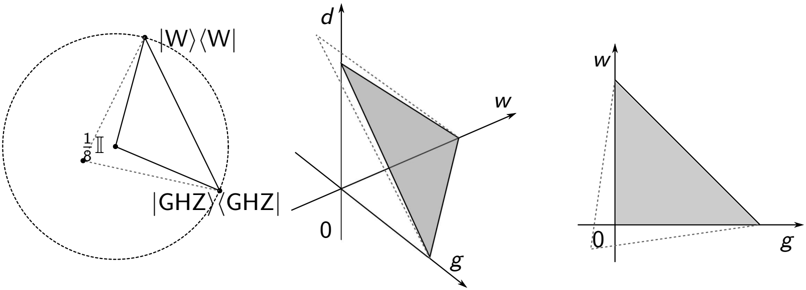
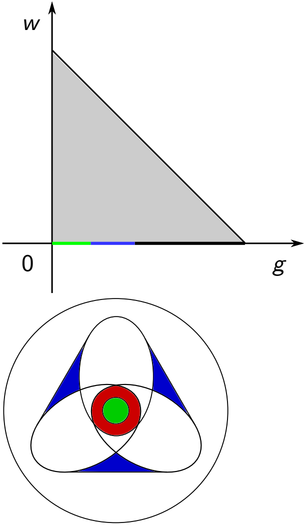


Exact results in literature



noisless GHZ-W mixture

- $d = 0$ line: measures of entanglement exactly calculated

noisy GHZ state

- $w = 0$ line:
- fully separable *iff* $0 \leq g \leq 1/5$
- for $1/5 < g \leq 3/7$ biseparable, yet inseparable under bipartitions,
- fully entangled $3/7 < g \leq 1$

Two-partite criteria. . .

. . . for three-partite system

- consider the 2nd and 3rd subsystems as one subsystem
- denote: $\mathcal{H} = \mathcal{H}^1 \otimes (\mathcal{H}^2 \otimes \mathcal{H}^3)$ as $\mathcal{H}^1 \otimes \mathcal{H}^{23}$
- recall: $\varrho \in \mathcal{D}$ is separable iff $\varrho = \sum_i p_i \varrho_i^1 \otimes \varrho_i^{23}$ decomp. exist
- either separable or entangled
- criteria for two-partite separability give criteria for $\mathcal{D}^{1-23}$

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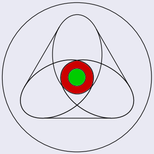
permutation invariant states



- $\mathcal{D}^{1-23} \cap \mathcal{D}^{2-13} \cap \mathcal{D}^{3-12}$
- union of fully separable and semiseparable states
- (convex set)

Entropy criterion

Rényi entropies

- $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$ probability distributions
- Rényi entropies:

$$H_\alpha(p) = \frac{1}{1-\alpha} \ln \sum_i p_i^\alpha \qquad \text{for} \qquad 0 \leq \alpha$$

- $\alpha = 0$ Hartley entropy: $H_0(p) = \ln |\{p_i \mid p_i \neq 0\}|$
- $\alpha \to 1$ Shanon entropy: $H_1(p) = \lim_{\alpha \to 1} H_\alpha(p) = -\sum_i p_i \ln p_i$
- $\alpha \to \infty$ Chebyshew entropy: $H_\infty(p) = \lim_{\alpha \to \infty} H_\alpha(p) = -\ln p_{\max}$

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Characterize the disorderness

- $H_\alpha(p) \geq H_\alpha(q)$: p is more disordered than q (for $\alpha \leq 1$ only!)
- $H_\alpha(\frac{1}{n}, \frac{1}{n}, \ldots) \geq H_\alpha(p) \geq H_\alpha(1, 0, \ldots)$

Entropy criterion II.

Quantum Rényi entropies

- Rényi entropies for density matrices:

$$S_\alpha(\varrho) = \frac{1}{1-\alpha} \ln \operatorname{Tr} \varrho^\alpha \qquad \text{for} \qquad 0 \leq \alpha$$

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Characterize the mixedness

- $S_\alpha(\varrho) \geq S_\alpha(\varrho')$: ϱ is more mixed than ϱ' (for $\alpha \leq 1$ only!)
- $S_\alpha(\frac{1}{\operatorname{Tr}\mathbb{I}}\mathbb{I}) \geq S_\alpha(\varrho) \geq S_\alpha(|\psi\rangle\langle\psi|)$

Entropy criterion III.

Entropy criterion

$$\varrho \quad \text{separable} \qquad \Longrightarrow \qquad S_\alpha(\varrho) \geq S_\alpha(\varrho^A) \quad \text{and} \quad S_\alpha(\varrho) \geq S_\alpha(\varrho^B)$$

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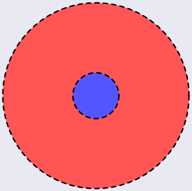
General scheme in the following



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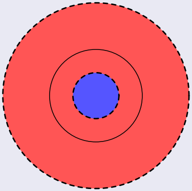

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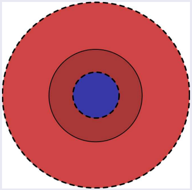

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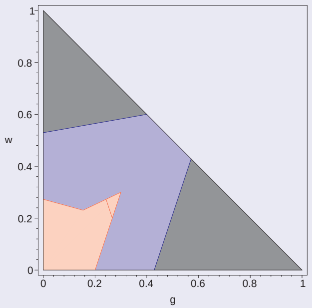
$\varrho \in$ subset $\Longrightarrow$ conditions hold

$\varrho \notin$ subset $\Longleftarrow$ cond. don't hold

Entropy criterion – results

Applying to noisy GHZ-W mixture



- $\alpha \to \infty$ (Chebyshew)
- red domain:
  $S_\infty(\varrho) \geq S_\infty(\varrho^1)$ and $S_\infty(\varrho) \geq S_\infty(\varrho^{23})$
- blue domain:
  $S_\infty(\varrho) \geq S_\infty(\varrho^1)$ and $S_\infty(\varrho) \ngeq S_\infty(\varrho^{23})$
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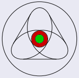
-  convex subset in red domain.

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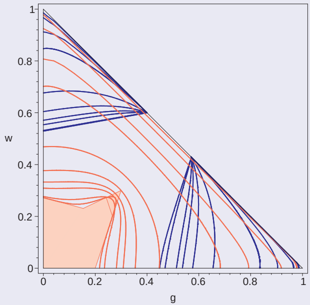
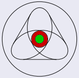


- $\alpha = \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, 2, 3, 4, 5, 10, 20$
- red curve: the border of the domain $S_\alpha(\varrho) \geq S_\alpha(\varrho^1)$ and $S_\alpha(\varrho) \geq S_\alpha(\varrho^{23})$
- blue curve: the border of the domain $S_\alpha(\varrho) \geq S_\alpha(\varrho^1)$ and $S_\alpha(\varrho) \ngeq S_\alpha(\varrho^{23})$
- red domain copied from $\alpha \to \infty$ case
-  convex subset in red domain.

Majorisation criterion I.

Majorisation

- $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$ probability distributions
- $p \prec q$ (q *majorize* p) if $\sum_{i=1}^{k} p_i^{\downarrow} \leq \sum_{i=1}^{k} q_i^{\downarrow}$ for all $1 \leq k \leq n$
- ($\downarrow$ denotes nonincreasing order)
- *partial order* on the set of probability distributions *up to permutations*
- "q is more ordered than p"

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Examples

- $(\frac{1}{n}, \frac{1}{n}, \dots) \prec p \prec (1, 0, \dots)$ for all p
- $p^{\downarrow} := (1/2, 1/8, \dots)$ and $q^{\downarrow} := (1/3, 1/3, \dots)$, then $p \not\prec q$ and $q \not\prec p$

Majorisation criterion II.

Majorisation criterion (Nielssen)

- majorisation for density matrices: $\omega \prec \sigma$ if Spect$(\omega) \prec$ Spect(σ)

$$\varrho \quad \text{separable} \qquad \Longrightarrow \qquad \varrho \prec \varrho^A \quad \text{and} \quad \varrho \prec \varrho^B$$

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Schur-concavity

- f real valued function on probability distributions
- $f(p)$ Schur-concav, if $p \prec q \Rightarrow f(p) \geq f(q)$
- entropies: Schur-concave functions

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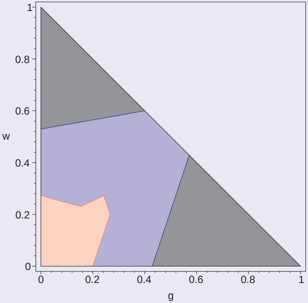
- entropy criterion can not be stronger than majorisation criterion

$$\varrho \quad \text{separable} \quad \Longrightarrow \quad \begin{matrix} \varrho \prec \varrho^A \\ \varrho \prec \varrho^B \end{matrix} \quad \Longrightarrow \quad \begin{matrix} S_\alpha(\varrho) \geq S_\alpha(\varrho^A) \\ S_\alpha(\varrho) \geq S_\alpha(\varrho^B) \end{matrix}$$

Majorisation criterion – results

Applying to noisy GHZ-W mixture



- red domain: $\varrho \prec \varrho^1$ and $\varrho \prec \varrho^{23}$
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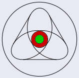
-  convex subset in red domain.

Partial transposition criterion I.

Partial transposition

- $(A \otimes B)^{T_1} = A^T \otimes B$ linear, trace-preserving
- recall: ϱ separable iff $\varrho = \sum_i p_i \varrho_i^1 \otimes \varrho_i^2$ decomposition exist
- then $\varrho^{T_1} = \sum_i p_i (\varrho_i^1)^T \otimes \varrho_i^2$ also a valid density operator

Partial transposition criterion I.

Partial transposition

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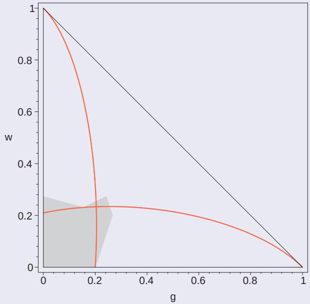
Partial transposition criterion (Peres, Horodecki)

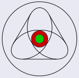
$$\varrho \quad \text{separable} \qquad \Longrightarrow \qquad \varrho^{T_1} \geq 0$$

- necessary and sufficient iff $\dim \mathcal{H} \leq 6$
- else: positive partial transposed entangled states (PPTES)
- PPTES: undistillable, hard to find

Partial transposition criterion – results

Applying to noisy GHZ-W mixture



- $\varrho^{T_1} \geq 0$ holds inside the intersection of the domains bounded by the red ellipses.
- grey domain: copied from majorisation criterion
-  convex subset in the intersection

Positive map criterion (Horodecki)

Positive map criterion

- Φ positive map, if preserves positivity of operators
- positive map criterion

$$\varrho \quad \text{separable} \qquad \Longleftrightarrow \qquad (\Phi \otimes \mathbb{I})\varrho \geq 0 \quad \text{for all positive maps } \Phi$$

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- necessary and sufficient, but hard to check for all positive maps
- for particular Φ: necessary but not sufficient criterion

$$\varrho \quad \text{separable} \qquad \Longrightarrow \qquad (\Phi \otimes \mathbb{I})\varrho \geq 0$$

- for $\Phi(A) = A^T$: partial transposition criterion
- for $\Phi(A) = \mathrm{Tr}(A)\mathbb{I} - A$: reduction criterion

Reduction criterion

Reduction map

- $\Phi(A) = \mathrm{Tr}(A)\mathbb{I} - A$
- $(\mathrm{Tr}(A)\mathbb{I} - A = (\sigma_2 A \sigma_2)^T$ for 2×2 matrices)

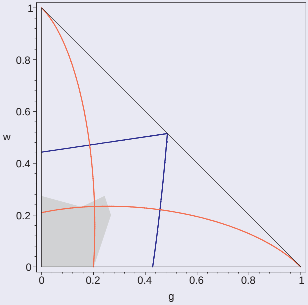
Reduction criterion (Horodecki)

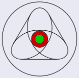
$$\varrho \quad \text{separable} \qquad \Longrightarrow \qquad \varrho^A \otimes \mathbb{I}^B - \varrho \geq 0 \quad \text{and} \quad \mathbb{I}^A \otimes \varrho^B - \varrho \geq 0$$

- can not be stronger than partial transposition crit.
- equivalent for qubit-qudit systems
- violated for distillable states
- there are no bound entangled 2×4 NPT states

Reduction criterion – results

Applying to noisy GHZ-W mixture



- reduction criterion holds inside the intersection of the domains bounded by the red and blue curves
  (two additional constraint (blue) are weaker)
- grey domain: copied from majorisation criterion
-  convex subset in the intersection

Reshuffling criterion (Horodecki)

Reshuffling map

- global map R
- on matrix elements: $[R(\varrho)]_{ii',jj'} = \varrho_{ij,i'j'}$
  (if $\varrho = \varrho_{ij;i'j'} |i\rangle\langle i'| \otimes |j\rangle\langle j'|$)

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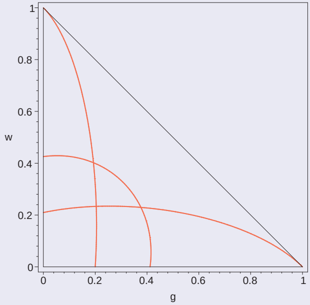
Reshuffling criterion

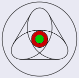
$$\varrho \quad \text{separable} \qquad \Longrightarrow \qquad \|R(\varrho)\|_{\mathrm{Tr}} \leq 1$$

- (trace norm $\|A\|_{\mathrm{Tr}} = \mathrm{Tr}\sqrt{A^\dagger A}$)

Reshuffling criterion – results

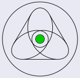
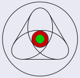
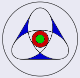
Applying to noisy GHZ-W mixture



- $\|R(\varrho)\|_{\mathrm{Tr}} \leq 1$ holds inside the red curve of high order
- red ellipses: copied from partial transposition criterion
- grey domain: copied from majorisation criterion
-  convex subset in the intersection

Multipartite criteria

criteria for three-partite separability classes

$\mathcal{D}^{3\text{-sep}}$  $\subset$ $\mathcal{D}^{1-23} \cap \mathcal{D}^{2-13} \cap \mathcal{D}^{3-12}$  $\subset$ $\mathcal{D}^{2\text{-sep}}$ 

Permutation criterion

Note that

- part. transp.: $[T_1(\varrho)]_{i'j,ij'} = \varrho_{ij,i'j'}$
- part. transp. crit.: ϱ separable $\Rightarrow$ $T_1(\varrho) \geq 0 \Leftrightarrow \|T_1(\varrho)\|_{\mathrm{Tr}} = 1$

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Permutation criterion for full separability (Horodecki)

- $\pi \in S_{2N}$ permutation, (N: number of subsystems)
- Λ_π map: $[\Lambda_\pi(\varrho)]_{i_{\pi(1)}i_{\pi(2)}\dots i_{\pi(N)},i_{\pi(N+1)}i_{\pi(N+2)}\dots i_{\pi(2N)}} = \varrho_{i_1 i_2\dots i_N,i_{N+1}i_{N+2}\dots i_{2N}}$

$$\varrho \quad \text{fully separable} \quad \Longrightarrow \quad \|\Lambda_\pi(\varrho)\|_{\mathrm{Tr}} \leq 1, \quad \forall \pi \in S_{2N}$$

Permutation criteria II.

Permutation criterion

- $(2N)!$ permutations give $(2N)!$ criteria for N-partite systems
- there are equivalent ones
- for two subsystems: two equivalence classes (representant elements are given by part. transp. and reshuffling map)
- for three subsystems: six equivalence classes of criteria (representant elements are given by three one-partite transpositions and three two-partite reshufflings)

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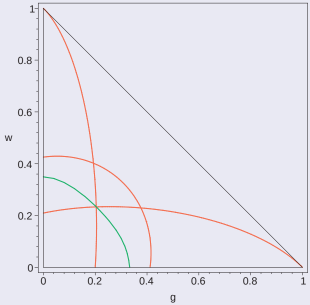
Two-partite reshuffling criterion for three-qubit system

- reshuffling of 2nd and 3rd subsystems: R'
- on matrix elements: $[R'(\varrho)]_{ijj',i'kk'} = \varrho_{ijk,i'j'k'}$

$$\varrho \quad \text{fully separable} \qquad \Longrightarrow \qquad \|R'(\varrho)\|_{\mathrm{Tr}} \leq 1$$

Permutation criteria – results

Applying to noisy GHZ-W mixture



- $\|R'(\varrho)\|_{\mathrm{Tr}} \leq 1$ holds inside the green curve of high order
- red curve of high order: copied from reshuffling criterion
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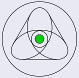
-  convex subset in the intersection

Criteria on spin-observables

Construction (Seevinck, Uffink)

- $(X^{(1)}, Y^{(1)}, Z^{(1)})$ orthogonal r.h. spin-observables on subsystems
- Pauli group: $[X^{(1)}, Y^{(1)}] = i2Z^{(1)}$
- $\langle X^{(1)}\rangle^2 + \langle Y^{(1)}\rangle^2 + \langle Z^{(1)}\rangle^2 \leq \langle \mathbb{I}^{(1)}\rangle^2$, equality *iff* the state is pure

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- obtain two sets of two-qubit observables:
  $(X_x^{(2)}, Y_x^{(2)}, Z_x^{(2)}, I_x^{(2)})$, where $x = 0, 1$
  e.g. $X_0^{(2)} := (X^{(1)} \otimes X^{(1)} - Y^{(1)} \otimes Y^{(1)})/2\dots$
- generalized Pauli group: $[X^{(2)}, Y^{(2)}] = i2Z^{(1)}$
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- obtain four sets of three-qubit observables:
  $(X_x^{(3)}, Y_x^{(3)}, Z_x^{(3)}, I_x^{(3)})$, where $x = 0\dots 3$
  e.g. $X_0^{(3)} := (X^{(1)} \otimes X_0^{(2)} - Y^{(1)} \otimes Y_0^{(2)})/2\dots$
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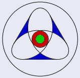
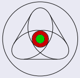
- $\langle X^{(3)}\rangle^2 + \langle Y^{(3)}\rangle^2 + \langle Z^{(3)}\rangle^2 \leq \langle I^{(3)}\rangle^2$, equality *iff* the state is pure
- can be generalized for N qubits: 2^N sets
- criteria by the expectation values $\langle X_x^{(N)}\rangle = \mathrm{Tr}\,\varrho X_x^{(N)} \dots$
- optimization on $(X^{(1)}, Y^{(1)}, Z^{(1)})$

Criteria on spin-observables

criteria by the expectation values

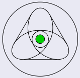


$$\varrho \in \mathcal{D}^{2-\mathrm{sep}} \implies \sqrt{\langle X_x^{(3)}\rangle^2 + \langle Y_x^{(3)}\rangle^2} \leq \sum_{y\neq x} \sqrt{\langle I_y^{(3)}\rangle^2 - \langle Z_y^{(3)}\rangle^2}$$



$$\varrho \in \mathcal{D}^{1-23} \cap \mathcal{D}^{2-13} \cap \mathcal{D}^{3-12} \implies$$

$$\max_x \left\{ \langle X_x^{(3)}\rangle^2 + \langle Y_x^{(3)}\rangle^2 \right\} \leq \min_x \left\{ \langle I_x^{(3)}\rangle^2 - \langle Z_x^{(3)}\rangle^2 \right\} \leq 1/4$$

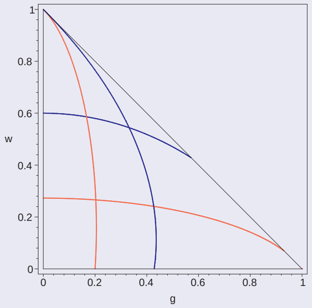


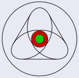
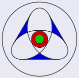
$$\varrho \in \mathcal{D}^{3-\mathrm{sep}} \implies$$

$$\max_x \left\{ \langle X_x^{(3)}\rangle^2 + \langle Y_x^{(3)}\rangle^2 \right\} \leq \min_x \left\{ \langle I_x^{(3)}\rangle^2 - \langle Z_x^{(3)}\rangle^2 \right\} \leq 1/16.$$

Criteria on spin-observables – results

Applying to noisy GHZ-W mixture



- Setting for the detection of GHZ: $(X^{(1)}, Y^{(1)}, Z^{(1)}) = (\sigma_x, \sigma_y, \sigma_z)$ for each subsystem
- Setting for the detection of W: $(X^{(1)}, Y^{(1)}, Z^{(1)}) = (\sigma_y, \sigma_z, \sigma_x)$ for each subsystem
-  convex subset in the intersection of the domains bounded by the red curves
-  convex subset in the intersection of the domains bounded by the blue curves

Criteria on matrix elements

basic idea (Gabriel et al)

- permutation operators acting on $\mathcal{H} \otimes \mathcal{H}$
- P_a: swaps the ath subsystems of the two copies:
  $P_a|i_1 \dots i_N j_1 \dots j_N\rangle = |i_1 \dots i_{a-1} j_a i_{a+1} \dots i_N j_1 \dots j_{a-1} i_a j_{a+1} \dots j_N\rangle$
- $P_S = \prod_{a\in S} P_a$ swaps an S set of subsystems
- if the ath subsystem is separable from the others then $P_a^\dagger \varrho^{\otimes 2} P_a = \varrho^{\otimes 2}$

$$\varrho \in \mathcal{D}^{k-\text{sep}} \quad \Longrightarrow$$

$$\sqrt{\langle\phi|\varrho^{\otimes 2} P_{\text{tot}}|\phi\rangle} \leq \sum_i \left(\prod_{n=1}^{k} \langle\phi|P^\dagger_{S_n^{(i)}} \varrho^{\otimes 2} P_{S_n^{(i)}}|\phi\rangle \right)^{\frac{1}{2k}}$$

where $\phi \in \mathcal{H} \otimes \mathcal{H}$ fully separable "detection vector",
i runs over possible k-partite splits $(S_1^{(i)}, S_2^{(i)}, \dots, S_k^{(i)})$

Criteria on matrix elements II.

results, notes

- optimization on $|\phi\rangle$ detection vectors
- $|\phi_{\text{GHZ}}\rangle = |000111\rangle$ detects GHZ (optimal?)
- $|\phi_{\text{W}}\rangle = H^{\otimes 6}|\phi_{\text{GHZ}}\rangle$ detects W (optimal?), where $H = 1/\sqrt{2}\left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right]$ Hadamard transf. for qubits

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- these gives the same results as the criteria on spin-observables
- moreover: $|\phi_{\mathrm{GHZ}}\rangle \mapsto |\phi_{\mathrm{W}}\rangle$ by Hadamard and $(\sigma_x, \sigma_y, \sigma_z) \mapsto (\sigma_z, -\sigma_y, \sigma_x) \equiv (\sigma_y, \sigma_z, \sigma_x)$ also by Hadamard $(\sigma_i \mapsto H\sigma_i H^\dagger)$
- we can consider Hadamard transformation on ϱ
- further connections between the two families of criteria?

Criteria on matrix elements

Different approach (Gühne, Seevinck)

- direct investigation of the matrix elements of pure separable states
- convexity argument $\Longrightarrow$ inequality for mixed states

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Example (for full separability)

- $|\Psi\rangle := (a_0|0\rangle + a_1|1\rangle) \otimes (b_0|0\rangle + b_1|1\rangle) \otimes (c_0|0\rangle + c_1|1\rangle)$,
  $\Psi_{ijk} = a_i b_j c_k$

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- $\varrho := |\Psi\rangle\langle\Psi|$ pure state, so $\varrho_{ijk,i'j'k'} = a_i b_j c_k \overline{a}_{i'} \overline{b}_{j'} \overline{c}_{k'}$

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- $\varrho_{000,000} = a_0 b_0 c_0 \overline{a}_0 \overline{b}_0 \overline{c}_0$
  $\varrho_{111,111} = a_1 b_1 c_1 \overline{a}_1 \overline{b}_1 \overline{c}_1$
  $\varrho_{000,000}\varrho_{111,111} = a_0 b_0 c_0 \overline{a}_1 \overline{b}_1 \overline{c}_1 \overline{a_0 b_0 c_0 \overline{a}_1 \overline{b}_1 \overline{c}_1} = \varrho_{000,111}\overline{\varrho_{000,111}}$
  for fully separable *pure* states $\varrho_{0,0}\varrho_{7,7} = |\varrho_{0,7}|^2$

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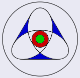
  for fully separable *pure* states $\varrho_{0,0}\varrho_{7,7} = |\varrho_{0,7}|^2$
- $\sqrt{fg}$ is concave, if f, g positive linear,
- $\sqrt{\varrho_{0,0}\varrho_{7,7}} - |\varrho_{000,111}| \geq 0$ for fully separable states

Criteria on matrix elements II.

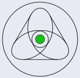
criteria obtained this way



$$\varrho \in \mathcal{D}^{2-\text{sep}} \quad \Longrightarrow$$

$$|\varrho_{07}| \leq \sqrt{\varrho_{66}\varrho_{11}} + \sqrt{\varrho_{55}\varrho_{22}} + \sqrt{\varrho_{33}\varrho_{44}},$$

$$|\varrho_{12}| + |\varrho_{14}| + |\varrho_{24}| \leq \sqrt{\varrho_{00}\varrho_{33}} + \sqrt{\varrho_{00}\varrho_{55}} + \sqrt{\varrho_{00}\varrho_{66}} + (\varrho_{11} + \varrho_{22} + \varrho_{44})/2.$$



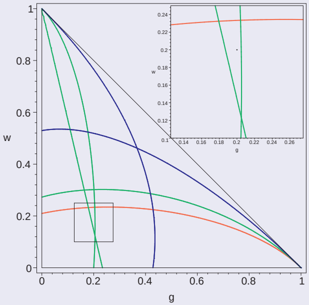
$$\varrho \in \mathcal{D}^{3-\text{sep}} \quad \Longrightarrow$$

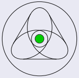
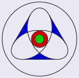
$$|\varrho_{07}| \leq (\varrho_{11}\varrho_{22}\varrho_{33}\varrho_{44}\varrho_{55}\varrho_{66})^{1/6},$$

$$|\varrho_{12}| + |\varrho_{14}| + |\varrho_{24}| \leq \sqrt{\varrho_{00}\varrho_{33}} + \sqrt{\varrho_{00}\varrho_{55}} + \sqrt{\varrho_{00}\varrho_{66}}.$$

Criteria on matrix elements – results

Applying to noisy GHZ-W mixture



-  convex subset in the intersection of the domains bounded by the green curves
-  convex subset in the intersection of the domains bounded by the blue curves
- red curve: copied from partial transpose criterion
- we have found a set of PPTES!

SLOCC-classes

Classification of pure states

- Stochastic Local Operation and Classical Communication
- $|\psi\rangle \sim |\psi'\rangle$ iff $|\psi'\rangle = G_1 \otimes \cdots \otimes G_N|\psi\rangle$, $G_i \in \mathrm{GL}(\mathcal{H}^i)$

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  "three qubits can be entangled in two inequivalent way"
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- fully entangled states:
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- repr. elements of SLOCC equivalence classes: $|\mathrm{GHZ}\rangle$ and $|\mathrm{W}\rangle$
- three-tangle τ_{123}:
  local SL-invariant, $\tau_{123}(\psi) \neq 0$ iff $|\psi\rangle$ in GHZ-class

SLOCC-classes II.

Mixed states

- $\mathcal{D}_3^{\mathrm{W}}$: mixture of projectors onto 2-separable and W-type vectors
- $\mathcal{D}^{3-\mathrm{sep}} \subset \mathcal{D}^{2-\mathrm{sep}} \subset \mathcal{D}^{\mathrm{W}} \subset \mathcal{D}^{\mathrm{GHZ}} \equiv \mathcal{D}$

SLOCC-classes II.

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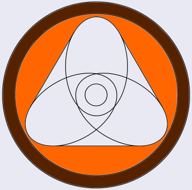

Classes:

- *Class W*: $\mathcal{D}^{\mathrm{W}} \setminus \mathcal{D}_3^{2-\mathrm{sep}}$
- *Class GHZ*: $\mathcal{D}^{\mathrm{GHZ}} \setminus \mathcal{D}_3^{\mathrm{W}}$
- fully entangled class $=$ Class W $\cup$ Class GHZ

SLOCC-classes II.

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- $\mathcal{D}_3^{\mathrm{W}}$: mixture of projectors onto 2-separable and W-type vectors
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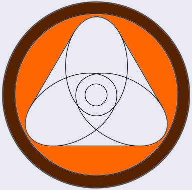

Classes:

- *Class W*: $\mathcal{D}^{\mathrm{W}} \setminus \mathcal{D}_3^{2-\mathrm{sep}}$
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- fully entangled class $=$ Class W $\cup$ Class GHZ

- "convex roof extension" of τ_{123}:

$$\tau_{123}(\varrho) = \min \left\{ \sum_i p_i \tau_{123}(\psi_i) \middle| \sum_i p_i |\psi_i\rangle\langle\psi_i| = \varrho \right\}$$

$\tau_{123}(\varrho) \neq 0$ iff $\varrho \in$ Class GHZ

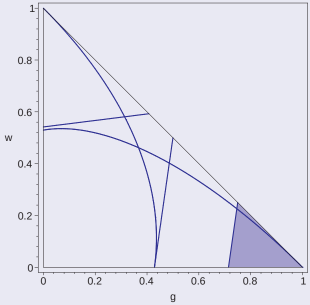
Witness operators

Detection of convex sets

- $\mathcal{C} \subset \mathcal{D}$ convex compact set,
- hermitian operator W is a *witness operator* for $\mathcal{C}$, if $\forall\sigma \in \mathcal{C}$: $0 \leq \operatorname{Tr} W\sigma$, and $\exists\varrho \notin \mathcal{C}$: $\operatorname{Tr} W\varrho < 0$
- W detects that $\varrho \notin \mathcal{C}$
- there exist witness for all $\varrho \notin \mathcal{C}$ (thm)
- follows to Bell-type inequalities

SLOCC-classes – witnesses

Applying to noisy GHZ-W mixture



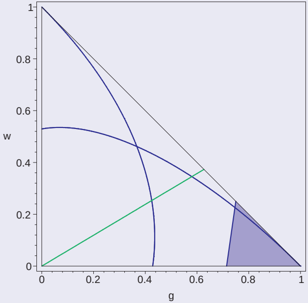
- blue straight lines: W-witnesses $\mathcal{W}_{W_1} = 2/3\mathbb{I} - |W\rangle\langle W|$ and $\mathcal{W}_{W_2} = 1/2\mathbb{I} - |GHZ\rangle\langle GHZ|$
- blue curves: criteria for biseparability of the previous section
- blue domain: all states here are of Class GHZ (using GHZ-witness $\mathcal{W}_{GHZ} = 3/4\mathbb{I} - |GHZ\rangle\langle GHZ|$)

SLOCC classes

GHZ-W mixture

- $d = 0$ line:
  $\tau_{123}(\varrho) = 0$ iff $0 \leq g \leq g_0 = 4 \cdot 2^{1/3}/(3 + 4 \cdot 2^{1/3}) = 0.626851\ldots$
- $\mathcal{D}^{\mathrm{W}}$ convex!

Applying to noisy GHZ-W mixture



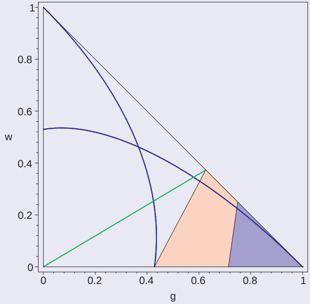
- state of Class-GHZ can not be above the green line

SLOCC classes

Noisy GHZ

- $w = 0$ line: fully entangled iff $3/7 < g$
- $\mathcal{D}^{\text{W}}$ convex!

Applying to noisy GHZ-W mixture



- blue domain: all states here are of Class GHZ
- the border of Class GHZ must be inside the red domain

Summary

Summary

- convex geometry of quantum states
- three-partite mixed state entanglement
- lots of criteria: different aspects of entanglement
- take a survey of an interesting subset of three-qubit states

Arising questions

- connections between criteria on spin-observables and detection vectors
- noisy GHZ state of arbitrary number of qubits
- multipartite criteria: seem to miss the point of entanglement
- . . .

Thank You For Your Attention!

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