Entanglement and correlations: an introduction

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10 and 17, October 2014

Outline

Quantum correlations

superposition principle: quantum systems behave nonclassically

- one single system: *uncertainty relations*
- composite systems: *nonclassical correlations* (discord, entanglement)
  even pure joint state may have mixed marginals

manybody systems: "physics of strongly correlated systems"

- correlation structure of (ground) states manifests itself
  also in macroscopic physical properties
- area law for correlations

fewbody systems: "quantum information theory"

- efficient q. algorithms, q. secure key sharing, q. teleportation
- "quantum correlation is a resource"

Our approach

discrete finite systems

- classical: configuration spaces of finite points (coin: 2, dice: 6,...)
- quantum: finite-dimensional Hilbert spaces
- geometrical "insight"
- the conceptual questions of quantum mechanics
  are not buried under hard problems of functional analysis :)
- still not a toy model!

quantum correlations as I like it

- of fundamental importance, beautiful, interesting and deep problems
- classical vs. quantum systems from information theoretical approach
- works in the lab too

Recall I. – States of a classical system

we know it certainly / all are the same: pure states

- $d < \infty$ mutually exclusive events:
  e.g. X prob. var. can take d different values $\mathbf{x} = (x_1, \dots, x_d)$
- d different pure *states*: $\mapsto \boldsymbol{\delta}_j = (0, \dots, 1, \dots, 0)$
  e.g., when $X = x_j$ with certainty
- expectation value is trivially $\langle X \rangle = x_j$ in pure state

we are uncertain / have an ensemble: mixed states

- different *pure states* $\boldsymbol{\delta}_j$, with p_j relative frequencies
- expectation value: $\langle X \rangle = \sum_j p_j x_j$
- probability density (*mixed state*): $\mathbf{p} = (p_1, \dots, p_d) = \sum_j p_j \boldsymbol{\delta}_j \in \Delta$
- after measuring X to be x_i, state *collapses*: $\mathbf{p} \mapsto \boldsymbol{\delta}_i$

Recall II. – States of a quantum system

we know it certainly / all are the same: pure states

- quantum system $\mapsto$ $\mathcal{H}$ *Hilbert space*, $d = \dim \mathcal{H} < \infty$
- *dyamical variables (observables):* its values are x_i, take $\{|\xi_i\rangle \in \mathcal{H}\}$ orthonormalized vectors, $X = \sum_i x_i |\xi_i\rangle\langle\xi_i| \in \mathrm{Lin}\,\mathcal{H}$ normal operator there exists noncommuting ones, $[X, Y] \neq 0$
- *state vectors:* $|\psi\rangle \in \mathcal{H}$, $(\|\psi\| = 1)$ then $|\psi\rangle = \sum_i \langle\xi_i|\psi\rangle|\xi_i\rangle$
- probability (!) of ith outcome *(Born's rule):* $q_i = |\langle\xi_i|\psi\rangle|^2$
- expectation value: $\langle X\rangle = \sum_j q_j x_j = \langle\psi|X|\psi\rangle$ nontrivial

we are uncertain / have an ensemble: mixed states

- different $|\psi_j\rangle \in \mathcal{H}$ state vectors, with p_j relative frequencies
- expectation value: $\langle X\rangle = \sum_j p_j \langle\psi_j|X|\psi_j\rangle = \mathrm{Tr}(\varrho X)$
- *density operator (mixed state):* $\varrho = \sum_j p_j |\psi_j\rangle\langle\psi_j| \in \mathcal{D} \subset \mathrm{Lin}_{\mathrm{SA}}\,\mathcal{H}$
- after measuring X to be x_i, state collapses $\varrho \mapsto |\xi_i\rangle\langle\xi_i|$

Recall III. – Quantum and classical "averages"

doing a measurement

- $X = \sum_i x_i |\xi_i\rangle\langle\xi_i|$ observable
- measurement statistics: $q_i = |\langle\xi_i|\psi\rangle|^2$, or $q_i = \mathrm{Tr}(\varrho|\xi_i\rangle\langle\xi_i|)$
- state collapses into the pure state $|\xi_i\rangle\langle\xi_i|$
- take a set $\{|\varphi_j\rangle \in \mathcal{H}\}$ of orthonormalized state vectors, and...

... in $\mathcal{H}$: linear combination (superposition)

- take $c_j \in \mathbb{C}$, $\|c\|_2 = 1$
  $|\varphi\rangle := \sum_j c_j |\varphi_j\rangle$
- measurement statistics:
  $q_i = |\sum_j c_j \langle\xi_i|\varphi_j\rangle|^2$
- interference!

... in $\mathcal{D}$: convex combination (mixture, "weighted average")

- take $0 \leq p_j \in \mathbb{R}$, $\|p\|_1 = 1$
  $\varrho := \sum_j p_j |\varphi_j\rangle\langle\varphi_j|$
- measurement statistics:
  $q_i = \sum_j p_j |\langle\xi_i|\varphi_j\rangle|^2$
- no interference!

Recall IV. – Classical "composite systems"

two observables in classical case

- two sets of mutually exclusive events (d_1, d_2):
  e.g. X and Y prob. vars. can take d_1 resp. d_2 different values
- $d_1 \times d_2$ different *pure states:* $\boldsymbol{\delta}_{12;ij} = (0,\dots,1,\dots,0) \in \mathbb{R}^{d_1} \otimes \mathbb{R}^{d_2}$
  e.g. $X = x_i$ and $Y = y_j$ with certainty
- different pure states $\boldsymbol{\delta}_{12;ij}$, with $p_{12;ij}$ relative frequencies, $\mapsto$ *joint prob. dens. (mixed state):* $\mathbf{p}_{12} = \sum_{ij} p_{12;ij} \boldsymbol{\delta}_{12;ij} \in \Delta_{12} \subset \mathbb{R}^{d_1} \otimes \mathbb{R}^{d_2}$
- *marginal state:* $\mathbf{p}_{12} \mapsto \mathbf{p}_2 = \mathrm{Sum}_1\, \mathbf{p}_{12}$, with $(\mathbf{p}_2)_j = p_{2,j} = \sum_i p_{12;ij}$
- after measuring X to be x_i, state collapses $\mathbf{p}_{12} \mapsto \mathbf{p}_{2|i}$:
  conditional state with $(\mathbf{p}_{2|i})_j = p_{12;ij}/p_{1;i}$ (Bayes')
- doesn't matter if the two sets of events (prob. vars.) correspond to
  - two different properties of *the same system*, or
  - (same or different) properties of *two different systems*

Recall V. – Quantum composite systems

two observables in quantum case

- *does* matter if the two sets of events (observables) correspond to
  - two different properties of *the same system*, or
  - (same or different) properties of *two different systems*
- in the former case, the observables usually $[X, Y] \neq 0$
- in the latter case, the observables $[X \otimes \mathbf{I}, \mathbf{I} \otimes Y] = 0$

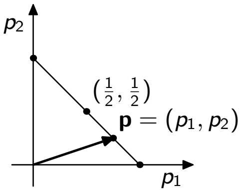
two subsystems

- two subsystems, $\mathcal{H}_1$, $\mathcal{H}_2$ Hilbert spaces, $d_a = \dim \mathcal{H}_a$
- *state vectors:* $|\psi_{12}\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 = \mathcal{H}_{12}$
- *mixed state:* $\varrho_{12} = \sum_i p_i |\psi_{12;i}\rangle\langle\psi_{12;i}| \in \mathcal{D}_{12} \subset \mathrm{Lin}_{\mathrm{SA}}\,\mathcal{H}_1 \otimes \mathrm{Lin}_{\mathrm{SA}}\,\mathcal{H}_2$
- *marginal state:* $\varrho_{12} \mapsto \varrho_2 = \mathrm{Tr}_1\,\varrho_{12}$, with $(\varrho_2)^{j}{}_{j'} = \sum_i \varrho^{i\,j}{}_{i\,j'}$,
- conditional state (of subsystem!): ill-defined in general,
  can only be defined with respect to the measurement

States of a system – Classical case

in general

- pure states: $\boldsymbol{\delta}_j = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^d$
- ensemble of systems in $\boldsymbol{\delta}_j$ with p_j relative frequencies $\mapsto$ mixed states: $\mathbf{p} = (p_1, \dots, p_d) = \sum_j p_j \boldsymbol{\delta}_j \in \Delta \subset \mathbb{R}^d$
- Δ simplex, the convex hull of the pure states: $\Delta = \mathrm{Conv}\{\boldsymbol{\delta}_j\}$
- finite number (d) of pure states, decomposition is unique!
- equivalently, $\Delta = \{\mathbf{p} \in \mathbb{R}^d \mid \mathbf{p} \geq 0,\ \mathrm{Sum}\,\mathbf{p} = 1\}$



example: bit ($d = 2$)

- pure states: $\boldsymbol{\delta}_1 = (1, 0)$, $\boldsymbol{\delta}_2 = (0, 1)$,
- states $\mathbf{p} = (p_1, p_2)$
- pure states: $p_1 = 1$ or $p_2 = 1$
- center: $(\frac{1}{2}, \frac{1}{2})$ "white noise"

States of a system – Classical case

in general

- pure states: $\boldsymbol{\delta}_j = (0,\dots,0,1,0,\dots,0) \in \mathbb{R}^d$
- ensemble of systems in $\boldsymbol{\delta}_j$ with p_j relative frequencies $\mapsto$ mixed states: $\mathbf{p} = (p_1,\dots,p_d) = \sum_j p_j \boldsymbol{\delta}_j \in \Delta \subset \mathbb{R}^d$
- Δ simplex, the convex hull of the pure states: $\Delta = \mathrm{Conv}\{\boldsymbol{\delta}_j\}$
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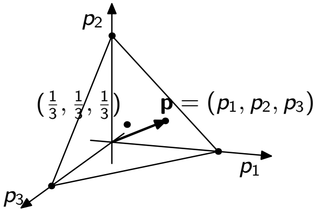


example: trit ($d = 3$)

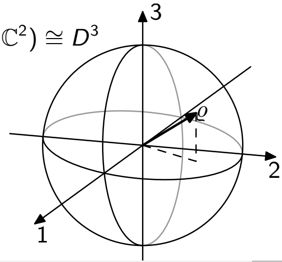
- pure states: $\boldsymbol{\delta}_1 = (1,0,0),\dots$
- states $\mathbf{p} = (p_1, p_2, p_3)$
- pure states: p_1 or p_2 or $p_3 = 1$
- center: $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ "white noise"

States of a system – Quantum case

in general

- pure states: $\pi = |\psi\rangle\langle\psi| \in \mathcal{P} \subset \mathrm{Lin}_{\mathrm{SA}}\,\mathcal{H}$ (geom.: $\mathcal{P} \cong \mathbb{C}\mathrm{P}^{d-1}$)
- ensemble of systems in π_j with p_j relative frequencies $\mapsto$
  mixed states: $\varrho = \sum_j p_j \pi_j \in \mathcal{D} \subset \mathrm{Lin}_{\mathrm{SA}}\,\mathcal{H}$ ($\mathcal{D} \subset \mathbb{R}^{d^2-1}$)
- $\mathcal{D}$ convex body, the convex hull of the pure states: $\mathcal{D} = \mathrm{Conv}\,\mathcal{P}$
- continuously many pure states, decomposition is not unique!
- equivalently, $\mathcal{D} = \{\varrho \in \mathrm{Lin}_{\mathrm{SA}}\,\mathcal{H} \mid \varrho \geq 0, \mathrm{Tr}\,\varrho = 1\}$

$\mathcal{D}(\mathbb{C}^2) \cong D^3$

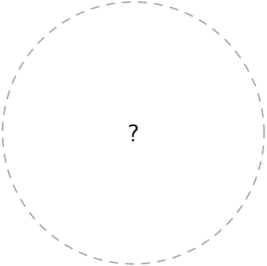


example: qubit ($d = 2$)

- $\mathcal{P}(\mathbb{C}^2) \cong \mathbb{C}\mathrm{P}^1 \cong S^2$: Bloch sphere
- $\mathbf{r}$ Bloch vector $\varrho = \frac{1}{2}(\mathbf{I} + \sum_\mu r_\mu \sigma_\mu)$
- pure st.: $|\mathbf{r}| = 1$, mixed st.: $|\mathbf{r}| < 1$
- center: $|\mathbf{r}| = 0$ "white noise"

States of a system – Quantum case

in general:
$\dim \mathcal{D} = d^2 - 1$,
$\dim \mathcal{P} = 2(d-1)$

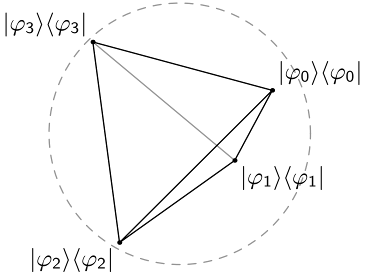


example: qudit ($d > 2$)

- set of pure states:
  $\mathcal{P} \cong \mathbb{C}\mathrm{P}^{d-1} \cong S^{2d-1}/S^1$
  not a sphere anymore
- but a subset (of zero measure) on the surface of a sphere,
  its center: white noise $\frac{1}{d}\mathbf{I}$
- set of states: $\mathcal{D} = \operatorname{Conv}\mathcal{P}$
- inside: $\operatorname{rk}\varrho = d$
- on the boundary: $\operatorname{rk}\varrho < d$
  (not necessarily pure states)
- pure states ($\mathcal{P}$): $\operatorname{rk}\varrho = 1$
  (extremal points)

States of a system – Quantum case

special: 3D-section containing four *orthogonal* pure states is a tetrahedron (simplex)

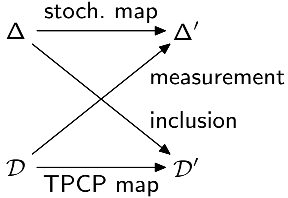


in general, intersection with a hyperplane is not even a polytope

example: qudit ($d > 2$)

- set of pure states: $\mathcal{P} \cong \mathbb{C}\mathrm{P}^{d-1} \cong S^{2d-1}/S^1$ not a sphere anymore
- but a subset (of zero measure) on the surface of a sphere, its center: white noise $\frac{1}{d}\mathbf{I}$
- set of states: $\mathcal{D} = \mathrm{Conv}\,\mathcal{P}$
- inside: $\mathrm{rk}\,\varrho = d$
- on the boundary: $\mathrm{rk}\,\varrho < d$ (not necessarily pure states)
- pure states ($\mathcal{P}$): $\mathrm{rk}\,\varrho = 1$ (extremal points)

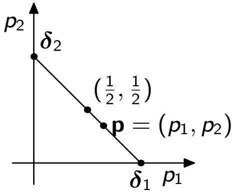
Maps of states – Overview



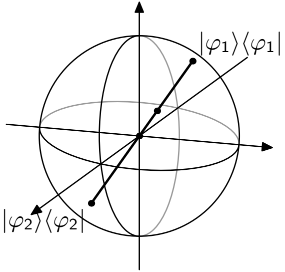
in general

- stochastic map: $\Delta \to \Delta'$
- TPCP map: $\mathcal{D} \to \mathcal{D}'$
- basis-dependent inclusion: $\Delta \to \mathcal{D}'$
- measurement (POVM): $\mathcal{D} \to \Delta'$

example: bit and qubit ($d = 2$)



$\longmapsto$



Transformations of states – Classical case

in general: classical channel

- recall: $\Delta = \{\mathbf{p} \in \mathbb{R}^d \mid \mathbf{p} \geq 0,\ \mathrm{Sum}\,\mathbf{p} = 1\}$
- $A : \Delta \to \Delta'$ map is a *stochastic map (Markov)*, i.e.

$$\mathbf{p} \longmapsto \mathbf{p}' = A(\mathbf{p}), \qquad A(\mathbf{p}) \geq 0,\ \mathrm{Sum}(A\mathbf{p}) = 1$$

- *A bistochastic* if stochastic and *unital:* from white noise it can make only white noise $A(\frac{1}{d}\mathbf{1}) = \frac{1}{d}\mathbf{1}$ ($d = d'$ enforced automatically)
- representation by *stochastic matrix* A: $A_{ij} \geq 0$, $\sum_i A_{ij} = 1$, if bistochastic then also $\sum_j A_{ij} = 1$

examples

- bit ($d = 2$): $A(t) = \left[\begin{smallmatrix} t & 1-t \\ 1-t & t \end{smallmatrix}\right]$ (also bistochastic)
- time evolution of a closed system: $A = R_\sigma$ permut. matrix ($\sigma \in S_d$)
- adding an uncorrelated ancilla, or dropping it

Transformations of states – Quantum case

in general: quantum channel

- recall: $\mathcal{D} = \{\varrho \in \mathrm{Lin}_{\mathrm{SA}}\,\mathcal{H} \mid \varrho \geq 0, \mathrm{Tr}\,\varrho = 1\}$
- $\mathcal{E} : \mathcal{D} \to \mathcal{D}'$ map is a *trace preserving complete positive map (TPCP)*
$$\varrho \longmapsto \varrho' = \mathcal{E}(\varrho), \qquad \mathcal{E}(\varrho) \geq 0,\ \mathrm{Tr}\,\mathcal{E}(\varrho) = 1,\ \mathcal{E} \otimes \mathcal{I}(\omega) \geq 0$$
- complete positivity: preserves the positivity of not only the system, but also the system and its (arbitrary) environment (quantum!)
- $\mathcal{E}$ *bistochastic* if stochastic and *unital:* from white noise it can make only white noise $\mathcal{E}(\frac{1}{d}\mathbf{I}) = \frac{1}{d}\mathbf{I}$ ($d = d'$ enforced automatically)
- Kraus representation: $\mathcal{E}(\varrho) = \sum_i K_i \varrho K_i^\dagger$, with $\sum_i K_i^\dagger K_i = \mathbf{I}$, if bistochastic then also $\sum_i K_i K_i^\dagger = \mathbf{I}$

examples

- time evolution of a closed system: $K = U$ unitary, $\mathcal{E}(\varrho) = U \varrho U^\dagger$
- adding an uncorrelated ancilla, or dropping it

Measurements – Classical case

in general

- observable: $\mathbf{x} = (x_1, \dots, x_d)$
- state: $\mathbf{p} = (p_1, \dots, p_d) = \sum_i p_i \boldsymbol{\delta}_i$
- observing x_i outcome: $\mathbf{p} \mapsto \mathbf{p}'_i = \boldsymbol{\delta}_i$ collapses, this is the result of a projection $P_i = \boldsymbol{\delta}_i \otimes \boldsymbol{\delta}_i^{\mathrm{T}}$

$$\mathbf{p} \quad \overset{\text{sel.}}{\longmapsto} \quad \left\{ \begin{array}{c} \mathbf{p}'_{(i)} = \dfrac{1}{q_{(i)}} P_i \mathbf{p} \equiv \boldsymbol{\delta}_i \\ q_{(i)} = \mathrm{Sum}\, P_i \mathbf{p} \equiv p_i \end{array} \right\} \quad \overset{\text{mix.}}{\longmapsto} \quad \mathbf{p}' = \sum_i q_{(i)} \mathbf{p}'_{(i)} \equiv \mathbf{p}$$

- non-selective measurement: doesn't disturb the state
- selective measurement: pure states aren't disturbed

Measurements – Quantum case

in general

- observable: $X = \sum_i x_i |\xi_i\rangle\langle\xi_i|$
- state: $\varrho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$
- observing x_i outcome: $\varrho \mapsto \varrho'_i = |\xi_i\rangle\langle\xi_i|$ collapses, this is the result of a projection $P_i(\cdot)P_i^\dagger = |\xi_i\rangle\langle\xi_i|(\cdot)|\xi_i\rangle\langle\xi_i|$

$$\varrho \quad \overset{\text{sel.}}{\longmapsto} \quad \left\{ \begin{aligned} \varrho'_{(i)} &= \frac{1}{q_{(i)}} P_i \varrho P_i^\dagger \equiv |\xi_i\rangle\langle\xi_i| \\ q_{(i)} &= \operatorname{Tr} P_i \varrho P_i^\dagger \equiv {\varrho^i}_i \end{aligned} \right\}$$

$$\overset{\text{mix.}}{\longmapsto} \quad \varrho' = \sum_i q_{(i)} \varrho'_{(i)} = \sum_i P_i \varrho P_i^\dagger \not\equiv \varrho$$

- even non-selective measurement disturbs the state
- even pure states are disturbed by selective measurement

Generalized measurements – Classical case

in general

- indirect projective measurements (meas. of an interacting ancilla)

$$\mathbf{p} \quad \overset{\text{sel.}}{\longmapsto} \quad \left\{ \begin{aligned} \mathbf{p}'_{(i)} &= \frac{1}{q_{(i)}} \operatorname{Sum}_{\text{Anc}}(\mathbf{1} \otimes P_i) R(\mathbf{p} \otimes \mathbf{p}_{\text{Anc}}) = \frac{1}{q_{(i)}} M_i \mathbf{p} \\ q_{(i)} &= \operatorname{Sum}(\mathbf{1} \otimes P_i) R(\mathbf{p} \otimes \mathbf{p}_{\text{Anc}}) = \operatorname{Sum} M_i \mathbf{p} \end{aligned} \right\}$$

$$\overset{\text{mix.}}{\longmapsto} \quad \mathbf{p}' = \sum_i q_{(i)} \mathbf{p}'_{(i)} = \operatorname{Sum}_{\text{Anc}} R(\mathbf{p} \otimes \mathbf{p}_{\text{Anc}}) = M\mathbf{p}$$

- outcomes, labelled by i, are given by *sum-non-increasing stochastic* maps M_i (*instrument*), for which $M = \sum_i M_i$ is (sum-preserving) stochastic
- even non-selective measurement disturbs the state
- even pure states are disturbed by selective measurement

Generalized measurements – Quantum case

in general

- indirect projective measurements (meas. of an interacting ancilla)

$$\varrho \overset{\text{sel.}}{\longmapsto} \left\{ \begin{aligned} \varrho'_{(i)} &= \frac{1}{q_{(i)}} \mathrm{Tr}_{\mathrm{Anc}} (\mathbf{I} \otimes P_i) U (\varrho \otimes \varrho_{\mathrm{Anc}}) U^\dagger (\mathbf{I} \otimes P_i)^\dagger = \frac{1}{q_{(i)}} \mathcal{M}_i(\varrho) \\ q_{(i)} &= \mathrm{Tr} (\mathbf{I} \otimes P_i) U (\varrho \otimes \varrho_{\mathrm{Anc}}) U^\dagger (\mathbf{I} \otimes P_i)^\dagger = \mathrm{Tr}\, \mathcal{M}_i(\varrho) \end{aligned} \right\}$$

$$\overset{\text{mix.}}{\longmapsto} \quad \varrho' = \sum_i q_{(i)} \varrho'_{(i)} = \mathrm{Tr}_{\mathrm{Anc}}\, U (\varrho \otimes \varrho_{\mathrm{Anc}}) U^\dagger = \mathcal{M}(\varrho)$$

- outcomes, labelled by i, are given by *trace-non-increasing CP* maps $\{\mathcal{M}_i\}$ (*instrument*), for which $\mathcal{M} = \sum_i \mathcal{M}_i$ is trace-preserving CP
- *Positive Operator Valued Measure (POVM):* $\{E_i = \sum_j M_{ij}^\dagger M_{ij} \geq 0\}$
- representation thm. (Naimark's): All such instrument $\{\mathcal{M}_i\}$ can be constructed by suitable ancilla with $\{P_i\}$, ϱ_{Anc} and U
- corollary: there are environmental representation of all $\mathcal{E}$ TPCP

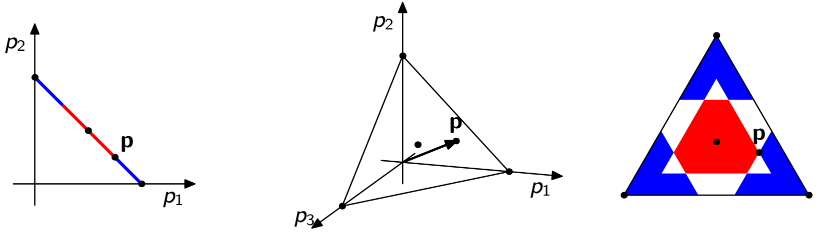
Mixedness by partial ordering – Classical case

in general

- *majorization* for classical states:

$$\mathbf{p} \preceq \mathbf{q} \quad \stackrel{\text{def.}}{\Longleftrightarrow} \quad \sum_{i=1}^{k} p_i^{\downarrow} \leq \sum_{i=1}^{k} q_i^{\downarrow} \quad \forall k = 1, 2, \dots, m,$$

- partial order, up to permutations, $\frac{1}{d}\mathbf{1} \preceq \mathbf{p} \preceq \boldsymbol{\delta}_1$



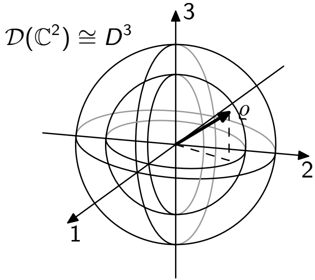
Mixedness by partial ordering – Quantum case

in general

- given ϱ: spectrum is the purest of any mixing weights, $\mathbf{p} \preceq \operatorname{Spect} \varrho$
- *majorization* for quantum states:

$$\varrho \preceq \omega \quad \overset{\text{def.}}{\Longleftrightarrow} \quad \operatorname{Spect} \varrho \preceq \operatorname{Spect} \omega$$

- partial order, up to unitaries, $\frac{1}{d}\mathbf{I} \preceq \varrho \preceq |\psi\rangle\langle\psi| = \pi$



example: qubit ($d = 2$)

- $\mathcal{P}(\mathbb{C}^2) \cong \mathbb{C}\mathrm{P}^1 \cong S^2$: Bloch sphere
- Bloch vector: $\varrho = \frac{1}{2}(\mathbf{I} + \sum_{i=1}^{3} r_i \sigma_i)$
- pure states: $|\mathbf{r}| = 1$
- center: $|\mathbf{r}| = 0$ "white noise"
- $\varrho \preceq \omega \quad \Longleftrightarrow \quad |\mathbf{r}_\varrho| \leq |\mathbf{r}_\omega|$

Mixedness by entropies – Classical case

in general

- mixedness: $f : \Delta \to \mathbb{R}$ *Schur-concave* function

$$\mathbf{p} \preceq \mathbf{q} \qquad \Longrightarrow \qquad f(\mathbf{p}) \geq f(\mathbf{q})$$

- *entropies:*

$$S(\mathbf{p}) = -\sum_i p_i \ln p_i, \qquad \text{Shannon entropy}$$

$$S^{\mathrm{R}}_\alpha(\mathbf{p}) = \frac{1}{1-\alpha} \ln \sum_i p_i^\alpha, \qquad \text{Rényi entropy}$$

$$S^{\mathrm{Ts}}_\alpha(\mathbf{p}) = \frac{1}{1-\alpha} \Big(\sum_i p_i^\alpha - 1 \Big), \qquad \text{Tsallis entropy}$$

- vanish exactly for pure states $\boldsymbol{\delta}_i$, taking maxima for white noise $\frac{1}{d}\mathbf{1}$
- Shannon's noiseless coding thm:
  Shannon entropy = information content

Mixedness by entropies – Quantum case

in general

- mixedness: $f : \mathcal{D} \to \mathbb{R}$ *Schur-concave* function
$$\varrho \preceq \omega \qquad \Longrightarrow \qquad f(\varrho) \geq f(\omega).$$
- given ϱ, spectrum has the lowest entr. $S(\varrho) := \min S(\mathbf{p}) = S(\operatorname{Spect}\varrho)$
- quantum entropies: entropies of the spectrum
$$S(\varrho) = -\operatorname{Tr}\varrho\ln\varrho, \qquad \text{von Neumann entropy}$$
$$S^{\mathrm{R}}_\alpha(\varrho) = \frac{1}{1-\alpha}\ln\operatorname{Tr}\varrho^\alpha, \qquad \text{quantum Rényi entropy}$$
$$S^{\mathrm{Ts}}_\alpha(\varrho) = \frac{1}{1-\alpha}\big(\operatorname{Tr}\varrho^\alpha - 1\big), \qquad \text{quantum Tsallis entropy}$$
- vanish exactly for pure states $|\psi\rangle\langle\psi|$, taking max. for white noise $\frac{1}{d}\mathbf{I}$
- Schumacher's noiseless coding thm:
von Neumann entropy = quantum information content

Distinguishability – Classical case

in general

- *relative entropy* of $\mathbf{p}, \mathbf{q} \in \Delta$ states
  $$D(\mathbf{p}||\mathbf{q}) = \sum_i p_i(\ln p_i - \ln q_i) \qquad \text{Kullback-Leibler divergence}$$
- there are Rényi, Tsallis generalizations too
- not symmetric, however, still has the most beautiful properties
- nonnegative, vanishes iff $\mathbf{p} = \mathbf{q}$
- Sanov's thm (hypothesis testing):
  relative entropy $=$ distinguishability

example

- in an experiment described by $\mathbf{q}$, the probability of that $\mathbf{p}$ is observed after finite n measurements goes $\sim \mathrm{e}^{-nD(\mathbf{p}||\mathbf{q})}$ for n large
- biased coin: $\mathbf{p}_{\text{biased}} = (1, 0)$, fair coin $\mathbf{p}_{\text{fair}} = (1/2, 1/2)$,
  $D(\mathbf{p}_{\text{biased}}||\mathbf{p}_{\text{fair}}) = \ln 2$, $D(\mathbf{p}_{\text{fair}}||\mathbf{p}_{\text{biased}}) = \infty$

Distinguishability – Quantum case

in general

- *quantum relative entropy* of $\varrho, \omega \in \mathcal{D}$ states
  $$D(\varrho||\omega) = \operatorname{Tr} \varrho(\ln \varrho - \ln \omega) \qquad \text{Umegaki relative entropy}$$
- ϱ and ω do not usually have common eigenbasis
- there are Rényi, Tsallis generalizations too
- not symmetric, however, still has the most beautiful properties
- nonnegative, vanishes iff $\varrho = \omega$
- quantum Stein's lemma (hypothesis testing):
  relative entropy = distinguishability
  (rate of decaying of the probability of confusing)

Mixedness and distinguishability – w.r.t. classical maps

compatibility with the notion of mixedness

- Hardy, Littlewood and Pólya's (HLP) lemma:
  bistochastic maps make states noisier

$$\mathbf{q} \preceq \mathbf{p} \quad \Longleftrightarrow \quad \exists A \text{ bistochastic, such that } \mathbf{q} = A(\mathbf{p}) \longleftarrow \mathbf{p}$$

- corollary: *entropies increase in bistochastic Markov chain*

$$A \text{ bistochastic} \quad \Longrightarrow \quad S\big(\mathbf{p}\big) \leq S\big(A(\mathbf{p})\big)$$

compatibility with the notion of distinguishability

- relative entropy is monotone decreasing under stochastic maps:

$$A \text{ stochastic} \quad \Longrightarrow \quad D\big(\mathbf{p}||\mathbf{q}\big) \geq D\big(A(\mathbf{p})||A(\mathbf{q})\big)$$

- *distinguishability decreases in Markov chains*
- note that $D(\mathbf{p}||\frac{1}{d}\mathbf{1}) = \ln d - S(\mathbf{p})$, so HLP follows

Mixedness and distinguishability – w.r.t. quantum maps

compatibility with the notion of mixedness

- quantum Hardy, Littlewood and Pólya's (qHLP) lemma:
  bistochastic maps make states noisier

$$\omega \preceq \varrho \quad \Longleftrightarrow \quad \exists\mathcal{E} \text{ bistochastic TPCP, such that } \omega = \mathcal{E}(\varrho) \longleftarrow\!\!\dashv \varrho$$

- corollary: *entropies increase in the chain of bistochastic TPCP*

$$\mathcal{E} \text{ bistochastic TPCP} \quad \Longrightarrow \quad S\big(\varrho\big) \leq S\big(\mathcal{E}(\varrho)\big)$$

compatibility with the notion of distinguishability

- quantum relative entropy is monotone decreasing under TPCP maps (proven by Lieb, Petz):

$$\mathcal{E} \text{ TPCP} \quad \Longrightarrow \quad D\big(\varrho||\omega\big) \geq D\big(\mathcal{E}(\varrho)||\mathcal{E}(\omega)\big)$$

- *distinguishability decreases in the chain of TPCP maps*
- note that $D(\varrho||\frac{1}{d}\mathbf{I}) = \ln d - S(\varrho)$, so qHLP follows

Mixedness and distinguishability – Overview

some abstractions

- the discussed monotonity properties seem to be the most important ones of classical and quantum entropies and relative entropies

$$\begin{aligned}
A \text{ stochastic} \quad &\Longrightarrow \quad D\big(\mathbf{p}||\mathbf{q}\big) \geq D\big(A(\mathbf{p})||A(\mathbf{q})\big)\\
A \text{ bistochastic} \quad &\Longrightarrow \quad S\big(\mathbf{p}\big) \leq S\big(A(\mathbf{p})\big)\\
\mathcal{E} \text{ TPCP} \quad &\Longrightarrow \quad D\big(\varrho||\omega\big) \geq D\big(\mathcal{E}(\varrho)||\mathcal{E}(\omega)\big)\\
\mathcal{E} \text{ bistochastic TPCP} \quad &\Longrightarrow \quad S\big(\varrho\big) \leq S\big(\mathcal{E}(\varrho)\big)
\end{aligned}$$

- *generalized* classical/quantum *entropies* and *relative entropies*, e.g. classical Tsallis/Rényi entropies and Tsallis/Rényi relative entropies; as well as the several extensions to the quantum case.
- moreover, let us stress that the notion of mixedness/distinguishability itself should be considered as a property which increases under bistochastic maps/decreases under stochastic maps

States of a bipartite system – Classical case

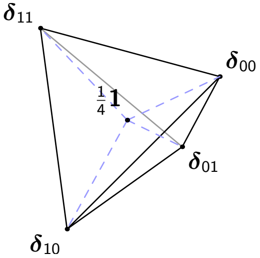
in general

- we have subsystems 1 and 2, with pure and mixed states
  $\boldsymbol{\delta}_{1;i} = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^{d_1}$, $\boldsymbol{\delta}_{2;j} = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^{d_2}$
  $\mathbf{p}_1 = \sum_i p_{1;i} \boldsymbol{\delta}_{1;i} \in \Delta_1 = \mathrm{Conv}\{\boldsymbol{\delta}_{1;i}\} \subset \mathbb{R}^{d_1}$,
  $\mathbf{p}_2 = \sum_j p_{1;j} \boldsymbol{\delta}_{1;j} \in \Delta_2 = \mathrm{Conv}\{\boldsymbol{\delta}_{2;j}\} \subset \mathbb{R}^{d_2}$
- pure states are always of the form $\boldsymbol{\delta}_{12;ij} = \boldsymbol{\delta}_{1;i} \otimes \boldsymbol{\delta}_{2;j} \in \mathbb{R}^{d_1} \otimes \mathbb{R}^{d_2}$
- mixed states: $\mathbf{p}_{12} = \sum_{ij} p_{12;ij} \boldsymbol{\delta}_{12;ij} \in \Delta_{12} = \mathrm{Conv}\{\boldsymbol{\delta}_{12;ij}\} \subset \mathbb{R}^{d_1} \otimes \mathbb{R}^{d_2}$
- decomposition is unique!
- equivalently, $\Delta_{12} = \{\mathbf{p}_{12} \in \mathbb{R}^{d_1} \otimes \mathbb{R}^{d_2} \mid \mathbf{p}_{12} \geq 0, \mathrm{Sum}\, \mathbf{p}_{12} = 1\}$

states of the subsystems

- *marginal state:* $\mathbf{p}_{12} \mapsto \mathbf{p}_2 = \mathrm{Sum}_1 \mathbf{p}_{12}$, with $(\mathbf{p}_2)_j = p_{2,j} = \sum_i p_{12;ij}$
- after measuring event i of subsys. 1, state of 2 collapses $\mathbf{p}_{12} \mapsto \mathbf{p}_{2|i}$:
  conditional state with $(\mathbf{p}_{2|i})_j = p_{12;ij} / p_{1;i}$ (Bayes')

States of a bipartite system – Classical case



example: two bits ($d_1 = d_2 = 2$)

- pure states:
  $$\boldsymbol{\delta}_{12;00} = (1,0) \otimes (1,0) = (1,0,0,0)$$
  $$\boldsymbol{\delta}_{12;01} = (1,0) \otimes (0,1) = (0,1,0,0)$$
  $$\boldsymbol{\delta}_{12;10} = (0,1) \otimes (1,0) = (0,0,1,0)$$
  $$\boldsymbol{\delta}_{12;11} = (0,1) \otimes (0,1) = (0,0,0,1)$$
- mixed states:
  $$\mathbf{p}_{12} = (p_{12;00}, p_{12;01}, p_{12;10}, p_{12;11})$$
- center: $(1/4, 1/4, 1/4, 1/4)$ "white noise"

States of a bipartite system – Quantum case

in general

- we have subsystems 1 and 2, with Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, with pure and mixed states
  $\pi_1 = |\psi_1\rangle\langle\psi_1| \in \mathcal{P}_1 \subset \mathrm{Lin}_{\mathrm{SA}}\,\mathcal{H}_1$, $\pi_2 = |\psi_2\rangle\langle\psi_2| \in \mathcal{P}_2 \subset \mathrm{Lin}_{\mathrm{SA}}\,\mathcal{H}_2$
  $\varrho_1 = \sum_i p_{1;i}\pi_{1;i} \in \mathcal{D}_1 = \mathrm{Conv}\,\mathcal{P}_1 \subset \mathrm{Lin}_{\mathrm{SA}}\,\mathcal{H}_1$,
  $\varrho_2 = \sum_i p_{1;i}\pi_{1;i} \in \mathcal{D}_2 = \mathrm{Conv}\,\mathcal{P}_2 \subset \mathrm{Lin}_{\mathrm{SA}}\,\mathcal{H}_2$
- pure states: $\pi_{12} = |\psi_{12}\rangle\langle\psi_{12}|$, are usually $\pi_{12} \neq \pi_1 \otimes \pi_2$
- mixed st.: $\varrho_{12} = \sum_i p_i \pi_{12;i} \in \mathcal{D}_{12} = \mathrm{Conv}\,\mathcal{P}_{12} \subset \mathrm{Lin}_{\mathrm{SA}}\,\mathcal{H}_1 \otimes \mathrm{Lin}_{\mathrm{SA}}\,\mathcal{H}_2$
- decomposition is not unique!
- equivalently, $\mathcal{D}_{12} = \{\varrho_{12} \in \mathrm{Lin}_{\mathrm{SA}}\,\mathcal{H}_1 \otimes \mathrm{Lin}_{\mathrm{SA}}\,\mathcal{H}_2 \mid \varrho_{12} \geq 0, \mathrm{Tr}\,\varrho_{12} = 1\}$

states of the subsystems

- *marginal state:* $\varrho_{12} \mapsto \varrho_2 = \mathrm{Tr}_1\,\varrho_{12}$, with $(\varrho_2)^{j}{}_{j'} = \sum_i (\varrho_{12})^{i\,j}{}_{i\,j'}$,
- *conditional state:* depends on measurement, we will see later

States of a bipartite system – Quantum case

example: mixed states of two qubits ($d_1 = d_2 = 2$)

- in Pauli basis $\{\mathbf{I}, \sigma_1, \sigma_2, \sigma_3\}$, coefficients $\mathbf{r}, \mathbf{s} \in \mathbb{R}^3$, $\mathbf{t} \in \mathbb{R}^3 \otimes \mathbb{R}^3$

$$\varrho_{12} = \frac{1}{4}\Big(\mathbf{I} \otimes \mathbf{I} + \sum_\mu r_\mu \sigma_\mu \otimes \mathbf{I} + \sum_\nu s_\nu \mathbf{I} \otimes \sigma_\nu + \sum_{\mu\nu} t_{\mu\nu} \sigma_\mu \otimes \sigma_\nu\Big)$$

- which parameters $\mathbf{r}, \mathbf{s}, \mathbf{t}$ lead to $\varrho_{12} \geq 0$?
- marginals (one qubit states, $\mathbf{r}, \mathbf{s}$ Bloch vectors):
  $\varrho_1 = \mathrm{Tr}_2\, \varrho_{12} = \frac{1}{2}(\mathbf{I} + \sum_\mu r_\mu \sigma_\mu)$, $\qquad \varrho_2 = \mathrm{Tr}_1\, \varrho_{12} = \frac{1}{2}(\mathbf{I} + \sum_\nu r_\nu \sigma_\nu)$
- special: *Pauli-diagonal* states, $\mathbf{r} = \mathbf{s} = \mathbf{0}$, $\mathbf{t} = \mathrm{diag}(t_1, t_2, t_3)$

$$\varrho_{12} = \frac{1}{4}\Big(\mathbf{I} \otimes \mathbf{I} + \sum_\mu t_\mu \sigma_\mu \otimes \sigma_\mu\Big)$$

- $\varrho_{12} \geq 0$ iff (t_1, t_2, t_3) lies in a tetrahedron (will see later)

States of a bipartite system – Quantum case: state vectors

Schmidt decomposition of state vectors

- let $\{|\varphi_{1;i}\rangle\}$ and $\{|\varphi_{2;j}\rangle\}$ bases in $\mathcal{H}_1$, $\mathcal{H}_2$
  state vector of bipartite system: $|\psi_{12}\rangle = \sum_{i,j=1}^{d_1,d_2} \psi_{12}^{ij} |\varphi_{1;i}\rangle \otimes |\varphi_{2;j}\rangle$
- based on the UDV-decomposition of matrices, by local unitary basis transf., $|\psi_{12}\rangle$ can be written in the LU-canonical form (Schmidt)

$$|\psi_{12}\rangle = \sum_{i=1}^{d_{\min}} \sqrt{\eta_i} |\varphi'_{1;i}\rangle \otimes |\varphi'_{2;i}\rangle$$

  with the *Schmidt coefficients* $\{\sqrt{\eta_i}\}$, with $\eta_i \geq 0$, $\sum_i \eta_i = \|\psi\|^2 = 1$
- the states of the subsystems in this basis:

$$\mathrm{Tr}_2\, \pi_{12} = \pi_1 = \sum_{i=1}^{d_{\min}} \eta_i |\varphi'_{1;i}\rangle\langle\varphi'_{1;i}| \qquad \mathrm{Tr}_1\, \pi_{12} = \pi_2 = \sum_{i=1}^{d_{\min}} \eta_i |\varphi'_{2;i}\rangle\langle\varphi'_{2;i}|$$

- so $\eta = \mathrm{Spect}\, \pi_1 = \mathrm{Spect}\, \pi_2$, and the *Schmidt rank:* $\mathrm{rk}\, \psi = \mathrm{rk}\, \pi_1$

States of a bipartite system – Quantum case: state vectors

examples: state vectors of two qubits ($d_1 = d_2 = 2$)

- let $\{|\varphi_{1;i}\rangle\}$ and $\{|\varphi_{2;j}\rangle\}$ bases in $\mathcal{H}_1$, $\mathcal{H}_2$
- Schmidt rank 1: e.g. $|00\rangle$, ($\equiv |\varphi_{1;0}\rangle \otimes |\varphi_{2;0}\rangle$ abbrev.) or $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
- Schmidt rank 2: e.g. *Bell states*

$$|\mathrm{B}_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad |\mathrm{B}_1\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

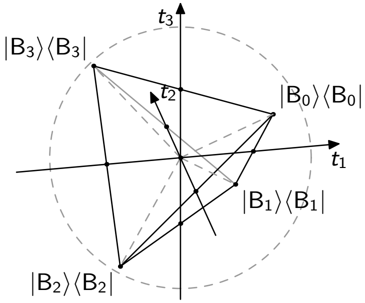
$$|\mathrm{B}_3\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \qquad |\mathrm{B}_2\rangle = \frac{-i}{\sqrt{2}}(|01\rangle - |10\rangle)$$

$$\pi_1 = \mathrm{Tr}_2(|\mathrm{B}_i\rangle\langle\mathrm{B}_i|) = \tfrac{1}{2}|0\rangle\langle 0| + \tfrac{1}{2}|1\rangle\langle 1| \sim \tfrac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

- in Schmidt form: $|\psi_\vartheta\rangle = \cos\vartheta|00\rangle + \sin\vartheta|11\rangle$, $0 \leq \vartheta \leq \pi/4$,

$$\pi_1 = \mathrm{Tr}_2(|\psi_\vartheta\rangle\langle\psi_\vartheta|) = \cos^2\vartheta|0\rangle\langle 0| + \sin^2\vartheta|1\rangle\langle 1| \sim \begin{bmatrix}\cos^2\vartheta & 0\\ 0 & \sin^2\vartheta\end{bmatrix}$$

States of a bipartite system – Quantum case

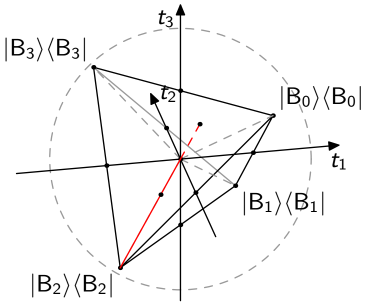


example: two qubits ($d_1 = d_2 = 2$)

- special: *Bell-diagonal states* $\varrho_{12} = \sum_i p_i |B_i\rangle\langle B_i|$
- it turns out: these are just the same as the *Pauli-diagonal states* (different parametrizations)

$$\varrho_{12} = \frac{1}{4}\Big(\mathbf{I}\otimes\mathbf{I} + \sum_\mu t_\mu \sigma_\mu \otimes \sigma_\mu\Big) = \sum_i p_i |B_i\rangle\langle B_i|$$

States of a bipartite system – Quantum case



example: two qubits ($d_1 = d_2 = 2$)

- special: *Bell-diagonal states*
  $\varrho_{12} = \sum_i p_i |B_i\rangle\langle B_i|$
- it turns out: these are just the same as the *Pauli-diagonal states* (different parametrizations)

$$\varrho_{12} = \frac{1}{4}\Big(\mathbf{I}\otimes\mathbf{I} + \sum_\mu t_\mu \sigma_\mu \otimes \sigma_\mu\Big) = \sum_i p_i |B_i\rangle\langle B_i|$$

- spec.spec.: *Werner states* (noisy Bell):

$$\varrho_{12} = w|B_2\rangle\langle B_2| + (1-w)\frac{1}{4}\mathbf{I}\otimes\mathbf{I}$$

for $-1/3 \leq w \leq 1$

Local maps of states – Overview

"global" maps of states

- classical case: $A : \Delta_{12} \to \Delta'_{12}$ stochastic maps + measurements
- quantum case: $\mathcal{E} : \mathcal{D}_{12} \to \mathcal{D}'_{12}$ TPCP maps + measurements

"local" maps of states: respecting the subsystem structure

- *Local Classical (LC):* stoch. maps+class meas. acting on a subsystem (sometimes a bit ill-defined in the quantum case, but useful if it's not)
- *Local Quantum (LQ):* TPCP maps+meas. acting on a subsystem

and we have also "communication"

- *Classical Communication (CC):* transferring classical information, e.g., in bits, that is, outcomes of local measurements (the modell of classical interaction among subsystems)
- *Quantum Communication (QC):* transferring quantum information, e.g., in qbits (the modell of quantum interaction among subsystems)

Local Quantum op. + Classical Communication = LQCC

example: teleportation (with three qubits, $d_1 = d_2 = d_3 = 2$)

- two *distant* laboratories (in the sense that QC is expensive)
- three subsystems with $\big(\mathcal{H}_1 \otimes \mathcal{H}_2\big) \otimes (\mathcal{H}_3)$, with the state vector $|\psi\rangle = |\chi\rangle \otimes |\mathrm{B}_0\rangle \equiv \frac{1}{2}\sum_i |\mathrm{B}_i\rangle \otimes \sigma_i|\chi\rangle$
- projective measurement in 12 subsys. $\{P_i = |\mathrm{B}_i\rangle\langle\mathrm{B}_i|\}$
- if measurement output is i then $|\psi'_{(i)}\rangle = |\mathrm{B}_i\rangle \otimes \sigma_i|\chi\rangle$, with $q_{(i)} = 1/4$
- output should be communicated to subsystem 3 (2 bits)
- then in subsystem 3, transformation $\sigma_i^{-1} = \sigma_i$ results in $|\mathrm{B}_i\rangle \otimes |\chi\rangle$
- the *shared* Bell state is used up (a resource)

Local Quantum op. + Classical Communication = LQCC

example: the simplest distillation protocoll (two qubits $d_1 = d_2 = 2$)

- two *distant* laboratories (in the sense that QC is expensive)
- shared systems of state vectors in $\mathcal{H}_1 \otimes \mathcal{H}_2$ in Schmidt form $|\psi\rangle = \sqrt{\eta_0}|00\rangle + \sqrt{\eta_1}|11\rangle$, with $\eta_0 \geq \eta_1 > 0$, $\eta_0 + \eta_1 = 1$
- we want to have $|\mathrm{B}_0\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)$
- first subsystem: measure with operators $\{M_0, M_1\}$
$$M_0 = \begin{bmatrix} \sqrt{\eta_1/\eta_0} & 0 \\ 0 & 1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} \sqrt{1-\eta_1/\eta_0} & 0 \\ 0 & 0 \end{bmatrix}$$
- if measurement output is 0 then $|\psi'_{(0)}\rangle = |\mathrm{B}_0\rangle$ (success)
  if measurement output is 1 then $|\psi'_{(1)}\rangle = |00\rangle$ (failure)
- output should be communicated to the second subsystem (1 bit)
- this is actually a *stochastic* LQ+CC (SLQCC):
  probability of success: $q_{(0)} = 1 - (\eta_0 - \eta_1)$, failure: $q_{(1)} = \eta_0 - \eta_1$

Local Quantum op. + Classical Communication = LQCC

results for bipartite pure states:

- *LQCC convertibility* (Nielsen's thm):
  $|\psi\rangle \longmapsto |\phi\rangle$ by LQCC iff $\boldsymbol{\eta}_\psi \preceq \boldsymbol{\eta}_\phi$
  (LQCC makes subsystems more pure)
- *LQCC interconvertibility:*
  $|\psi\rangle \longmapsto |\phi\rangle$ and $|\psi\rangle \longleftarrow |\phi\rangle$ by LQCC
  iff $|\phi\rangle = U_1 \otimes U_2|\psi\rangle$ with U_1, U_2 unitaries
  (they can be used for the same task)
- *SLQCC interconvertibility (stochastical):*
  $|\psi\rangle \longmapsto |\phi\rangle$ and $|\psi\rangle \longleftarrow |\phi\rangle$ by LQCC with nonzero prob. of success
  iff $|\phi\rangle = G_1 \otimes G_2|\psi\rangle$ with G_1, G_2 invertible
  (they can be used for the same task, with different prob. of success)
- for mixed states, the problems are still open

Recall – Correlations of observables vs. of states

usual statistical quantities

- *covariance* of two probabilistic variables: $\mathrm{COV}(X,Y) = \langle (X - \langle X\rangle)(Y - \langle Y\rangle)\rangle = \langle XY\rangle - \langle X\rangle\langle Y\rangle$
- *correlation* is a normalized version of this: $-1 \leq \mathrm{CORR}(X,Y) = \mathrm{COV}(X,Y)/\sqrt{\mathrm{COV}(X,X)\,\mathrm{COV}(Y,Y)} \leq 1$

more essential: correlations of states

- classical: $\mathrm{COV}(X,Y) = \sum_{ij}(p_{12;ij} - p_{1;i}p_{2;j})x_i y_j$
  correlation of the *events* (meas. outcomes) $C_{ij} = p_{12;ij} - p_{1;i}p_{2;j}$
  correlation "in the *state* itself:" $\mathbf{C} := \mathbf{p}_{12} - \mathbf{p}_1 \otimes \mathbf{p}_2$
  then $\mathrm{COV}(X,Y) = \mathbf{C}^{\mathrm{T}}\mathbf{x}\otimes\mathbf{y}$
- quantum: correlation of the *state* itself: $\Gamma := \varrho_{12} - \varrho_1 \otimes \varrho_2$
  then $\mathrm{COV}(X,Y) = \mathrm{Tr}\,\Gamma^{\mathrm{T}} X \otimes Y$
- in q.m. there are many (nontrivially) different observables in a system
- $\mathbf{C}$ and Γ remain meaningful even if there are no values, only events

Correlations – Classical case

classical case: uncorrelated / correlated

- correlation in the states is characterized by $\mathbf{C} := \mathbf{p}_{12} - \mathbf{p}_1 \otimes \mathbf{p}_2$
- events i and j are *uncorrelated* iff $p_{12;ij} = p_{1;i} p_{2;j}$, that is, $C_{ij} = 0$
- state is *uncorrelated* ($\mathbf{p}_{12} \in \Delta_{\text{uncorr}}$) iff $\mathbf{p}_{12} = \mathbf{p}_1 \otimes \mathbf{p}_2$, that is, $\mathbf{C} = \mathbf{0}$ (iff $\langle XY \rangle = \langle X \rangle \langle Y \rangle$ for all observables X, Y)
- else it is *correlated* ($\mathbf{p}_{12} \in \Delta_{12} \setminus \Delta_{\text{uncorr}}$)

uncorrelated states

- pure states are $\boldsymbol{\delta}_{12;ij} = \boldsymbol{\delta}_{1;i} \otimes \boldsymbol{\delta}_{2;j}$, automatically uncorrelated
- all states are mixtures of pure (then uncorrelated) states (uniquely), uncorrelated states are mixtures by product mixing weights

(a bit tautologic, but helps the quantum analogy)

selective measurement

- selective measurement on a subsystem disturbs the state of the other iff the state is correlated

Correlations – Classical case

classical case: uncorrelated / correlated

- correlation in the states is characterized by $\mathbf{C} := \mathbf{p}_{12} - \mathbf{p}_1 \otimes \mathbf{p}_2$
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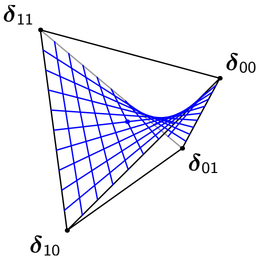

example: two bits ($d_1 = d_2 = 2$)

- pure states: $\boldsymbol{\delta}_{12;00} = (1,0) \otimes (1,0), \dots$
- mixed states:
  $\mathbf{p}_{12} = (p_{12;00}, p_{12;01}, p_{12;10}, p_{12;11})$
- uncorrelated states: $p_{12;ij} = p_{1;i} p_{2;j}$ iff
  $p_{12;00} p_{12;11} = p_{12;01} p_{12;10}$

Correlations – Quantum case I.: correlation

quantum case I: uncorrelated / correlated

- correlation in the states is characterized by $\Gamma := \varrho_{12} - \varrho_1 \otimes \varrho_2$
- state is *uncorrelated* ($\varrho_{12} \in \mathcal{D}_{\text{uncorr}}$) iff $\varrho_{12} = \varrho_1 \otimes \varrho_2$, that is, $\Gamma = 0$ (iff $\langle XY \rangle = \langle X \rangle \langle Y \rangle$ for all observables X, Y)
- then we say that the *two subsystems are uncorrelated*
- else it is *correlated* ($\varrho_{12} \in \mathcal{D}_{12} \setminus \mathcal{D}_{\text{uncorr}}$)

pure states

- pure states are not uncorrelated automatically! $\pi_{12} \neq \pi_1 \otimes \pi_2$,
  if a pure state is correlated, then the correlation is of quantum origin
- all states are mixtures of pure states (not uniquely), uncorrelated states are mixtures of pure uncorr. states by product mixing weights

selective measurement

- selective measurement on a subsystem disturbes the state of the other iff the state is correlated

Correlations – Quantum case I.: correlation

quantum case I: uncorrelated / correlated

- correlation in the states is characterized by $\Gamma := \varrho_{12} - \varrho_1 \otimes \varrho_2$
- state is *uncorrelated* ($\varrho_{12} \in \mathcal{D}_{\text{uncorr}}$) iff $\varrho_{12} = \varrho_1 \otimes \varrho_2$, that is, $\Gamma = 0$ (iff $\langle XY \rangle = \langle X \rangle \langle Y \rangle$ for all observables X, Y)
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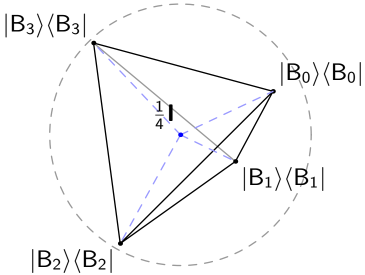


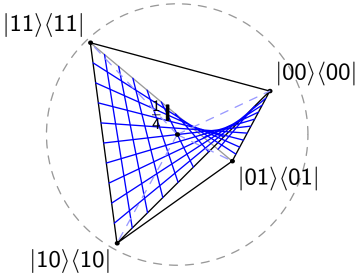
example: Bell-diag. states ($d_1 = d_2 = 2$)

- a special section of the whole $\mathcal{D}_{12}$
- pure states: $|\mathrm{B}_i\rangle\langle \mathrm{B}_i|$
- center: $\frac{1}{4}\mathbf{I}$ "white noise"
- uncorrelated states: the white noise

Correlations – Quantum case I.: correlation

quantum case I: uncorrelated / correlated

- correlation in the states is characterized by $\Gamma := \varrho_{12} - \varrho_1 \otimes \varrho_2$
- state is *uncorrelated* ($\varrho_{12} \in \mathcal{D}_{\text{uncorr}}$) iff $\varrho_{12} = \varrho_1 \otimes \varrho_2$, that is, $\Gamma = 0$ (iff $\langle XY \rangle = \langle X \rangle \langle Y \rangle$ for all observables X, Y)
- then we say that the *two subsystems are uncorrelated*
- else it is *correlated* ($\varrho_{12} \in \mathcal{D}_{12} \setminus \mathcal{D}_{\text{uncorr}}$)



example: embedded classical ($d_1 = d_2 = 2$)

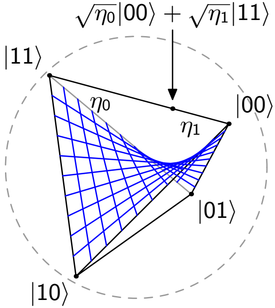
- a special section of the whole $\mathcal{D}_{12}$
- pure states: $|ij\rangle\langle ij|$ uncorrelated
- mixed states: $\sum_{ij} p_{ij} |ij\rangle\langle ij|$
- center: $\frac{1}{4}\mathbf{I}$ "white noise"
- uncorr.: $p_{ij} = p_i p_j$ iff $p_{00} p_{11} = p_{01} p_{10}$

Correlations – Quantum case: pure states (a detour)

by Schmidt decomposition of state vectors

- (I) pure states are not uncorrelated automatically! $\pi_{12} \neq \pi_1 \otimes \pi_2$, *if a pure state is correlated, then the correlation is of quantum origin*
- pure state: $\pi_{12} = |\psi_{12}\rangle\langle\psi_{12}|$, marginals: $\pi_1 = \mathrm{Tr}_2\, \pi_{12}$, $\pi_2 = \mathrm{Tr}_1\, \pi_{12}$
- Schmidt-canonical form: $|\psi_{12}\rangle = \sqrt{\eta_1}|11\rangle + \sqrt{\eta_2}|22\rangle + \cdots + \sqrt{\eta_d}|dd\rangle$
- (II) marginals are not necessary pure since $\mathrm{Spect}\, \pi_1 = \mathrm{Spect}\, \pi_2 = \boldsymbol{\eta}$ *"the best possible knowledge of the whole does not involve the best possible knowledge of its parts"* (Schrödinger)
- uncorrelated states: $\pi_{12} = \pi_1 \otimes \pi_2$ iff $|\psi_{12}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$
- or, π_{12} uncorrelated iff π_1 and π_2 are pure ($\boldsymbol{\eta}$ pure),

Correlations – Quantum case: pure states (a detour)



example: two qubit pure sts. ($d_1 = d_2 = 2$)

- two qubit state vectors $|\psi_{12}\rangle = \psi_{12}^{00}|00\rangle + \psi_{12}^{01}|01\rangle + \psi_{12}^{10}|10\rangle + \psi_{12}^{11}|11\rangle$
  spec: $\psi_{12}^{ij} \geq 0$
- uncorrelated states: $\psi_{12}^{ij} = \psi_1^i \psi_2^j$ iff
  $\psi_{12}^{00}\psi_{12}^{11} = \psi_{12}^{01}\psi_{12}^{10}$
- spec.spec.: Schmidt form:
  $|\psi_{12}\rangle = \sqrt{\eta_0}|00\rangle + \sqrt{\eta_1}|11\rangle$

Correlations – Quantum case II.: discord

quantum case II: non-discordant ("classical") / discordant ("non-classical")

- *local* inclusion of classical states into quantum ones: fixing *local* bases $\{|\varphi_{1;i}\rangle\}$, $\{|\varphi_{2;i}\rangle\}$, for pure states $\boldsymbol{\delta}_{1;i} \mapsto |\varphi_{1;i}\rangle\langle\varphi_{1;i}|$
- state is *non-discordant* ($\varrho_{12} \in \mathcal{D}_{\text{nondsc}}$) if it's an image of a class. one: $\varrho_{12} = \sum_{ij} p_{ij}\pi_{1;i} \otimes \pi_{2;j}$ with $\{\pi_{1;i}\}$, $\{\pi_{2;i}\}$ orthogonal
- else it is discordant ($\varrho_{12} \in \mathcal{D}_{12} \setminus \mathcal{D}_{\text{nondsc}}$)
- if uncorr. then nondisc. $\mathcal{D}_{\text{uncorr}} \subset \mathcal{D}_{\text{nondsc}}$, pure st. $\mathcal{P}_{\text{uncorr}} = \mathcal{P}_{\text{nondsc}}$

selective measurement

- selective measurement on a subsystem disturbes the state of the other iff the state is correlated
- for nondiscordant states: one *can find* local *nonselective* measurement which doesn't disturb the system
- can be diagonalized by local unitaries, $U_1 \otimes U_2 \varrho_{12} U_1^\dagger \otimes U_2^\dagger$ diagonal

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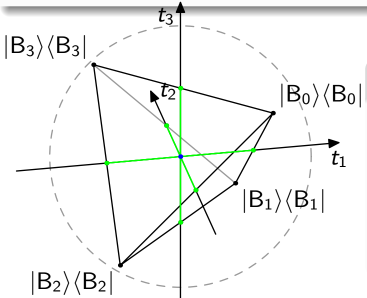

example: Bell-diag. states $(d_1 = d_2 = 2)$

- a special section of the whole $\mathcal{D}_{12}$, pure states: $|B_i\rangle\langle B_i|$
- uncorrelated states: white noise only
- nondisc.: $\varrho_{12} = \frac{1}{4}(\mathbf{I} \otimes \mathbf{I} + t_\mu \sigma_\mu \otimes \sigma_\mu)$

Correlations – Quantum case III.: entanglement

quantum case III: separable / entangled

- in the classical case: all states are mixtures of uncorrelated states
- state is *separable:* $\varrho_{12} \in \mathcal{D}_{\text{sep}}$ if it is the mixture of uncorrelated states (Werner): $\varrho_{12} = \sum_k p_k \varrho_{1;k} \otimes \varrho_{2;k}$
- else it is *entangled* ($\varrho_{12} \in \mathcal{D}_{12} \setminus \mathcal{D}_{\text{sep}}$) (decision of this is difficult)
- not entirely nondiscordant, $\mathcal{D}_{\text{nondsc}} \subset \mathcal{D}_{\text{sep}}$, pure states $\mathcal{P}_{\text{nondsc}} = \mathcal{P}_{\text{sep}}$

convexity

- states: $\mathcal{D} = \operatorname{Conv} \mathcal{P}$ convex hull of pure states
- separable states: $\mathcal{D}_{\text{sep}} = \operatorname{Conv} \mathcal{D}_{\text{uncorr}}$ convex hull of uncorr. states
- extremal points: pure states (there are separable and entangled ones) separable states can also be written as $\varrho_{12} = \sum_l q_l \pi_{l,1} \otimes \pi_{l,2}$
- separable states: $\mathcal{D}_{\text{sep}} = \operatorname{Conv} \mathcal{P}_{\text{sep}}$, convex hull of sep. pure states

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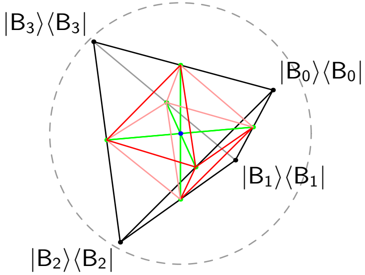

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- uncorrelated states: white noise only
- nondisc.: $\varrho_{12} = \frac{1}{4}(\mathbf{I} \otimes \mathbf{I} + t_\mu \sigma_\mu \otimes \sigma_\mu)$
- separable states: octahedron (PPT!)

Quantum correlations – Overview

definitions

- uncorr.: $\varrho_{12} = \varrho_1 \otimes \varrho_2 = \sum_{ij} p_i p_j \pi_{1;i} \otimes \pi_{2;j}$, $\{\pi_{a;i}\}$ orthogonal
- nondisc.: $\varrho_{12} = \sum_{ij} p_{ij} \pi_{1;i} \otimes \pi_{2;j}$, $\{\pi_{a;i}\}$ orthogonal
- sep.: $\varrho_{12} = \sum_k p_k \varrho_{1;k} \otimes \varrho_{2;k} = \sum_l q_l \pi_{1;l} \otimes \pi_{2;l}$, $\{\pi_{a;i}\}$ general

nested structure

- in general, $\mathcal{D}_{\text{uncorr}} \subset \mathcal{D}_{\text{nondsc}} \subset \mathcal{D}_{\text{sep}}$, that is,

  uncorrelated $\Longrightarrow$ nondiscordant ("classical") $\Longrightarrow$ separable

  correlated $\Longleftarrow$ discordant ("nonclassical") $\Longleftarrow$ entangled

- specially, for pure states, $\mathcal{P}_{\text{uncorr}} = \mathcal{P}_{\text{nondsc}} = \mathcal{P}_{\text{sep}}$, that is,

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Quantum correlations – Overview

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geometry

- in general, $\mathcal{D}_{\text{uncorr}} \subset \mathcal{D}_{\text{nondsc}} \subset \mathcal{D}_{\text{sep}} \subset \mathcal{D}_{12}$
- $\mathcal{D}_{\text{sep}} = \operatorname{Conv} \mathcal{P}_{\text{sep}}$ convex set, of nonzero measure in $\mathcal{D}_{12} = \operatorname{Conv} \mathcal{P}_{12}$
  $\mathcal{D}_{\text{nondsc}}$ is of zero measure in $\mathcal{D}_{\text{sep}}$,
  $\mathcal{D}_{\text{uncorr}}$ is of zero measure in $\mathcal{D}_{\text{nondsc}}$.
- specially, for pure states, $\mathcal{P}_{\text{uncorr}} = \mathcal{P}_{\text{nondsc}} = \mathcal{P}_{\text{sep}} \subset \mathcal{P}_{12}$
- $\mathcal{P}_{\text{uncorr}} = \mathcal{P}_{\text{nondsc}} = \mathcal{P}_{\text{sep}}$ is of zero measure in $\mathcal{P}_{12}$,

Quantum correlations – w.r.t. quantum maps

definitions

- uncorr.: $\varrho_{12} = \varrho_1 \otimes \varrho_2 = \sum_{ij} p_i p_j \pi_{1;i} \otimes \pi_{2;j}$, $\{\pi_{a;i}\}$ orthogonal
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creation

- all *uncorrelated* states can be created by *LC* from pure product state (assuming that LC is w.r.t. the local pure states)
- all *nondisc.* states can be created by *LC+CC* from pure product state (or from uncorrelated state)
  (assuming that LC is w.r.t. the local pure states)
- all *separable* states can be created by *LQ+CC* from pure product state (ultimate definition, in accordance with the distant lab paradigm)

Correlation measures – Classical case

classical case: correlation (measure)

- correlation in the state is characterized by $\mathbf{C} = \mathbf{p}_{12} - \mathbf{p}_1 \otimes \mathbf{p}_2$
- let the measure of corr. be the distinguishability of $\mathbf{p}_{12}$ and $\mathbf{p}_1 \otimes \mathbf{p}_2$:

$$D(\mathbf{p}_{12} || \mathbf{p}_1 \otimes \mathbf{p}_2) = S(\mathbf{p}_1) + S(\mathbf{p}_2) - S(\mathbf{p}_{12}) = I(\mathbf{p}_{12})$$

  this turns out to be the *mutual information* $I(\mathbf{p}_{12})$

mutual information

- vanishes exactly for uncorrelated states
- another (original?) definition: $J(\mathbf{p}_{12}) := S(\mathbf{p}_2) - S_{2|1}(\mathbf{p}_{12}) \equiv I(\mathbf{p}_{12})$
  with the conditional entropy $S_{2|1}(\mathbf{p}_{12}) = \sum_i p_i S(\mathbf{p}_{2|i})$
  with the entropy of the conditional state $\mathbf{p}_{2|i}$
- meaning: information gain about the subsystem measuring the other (this is symmetric in the classical case)

Correlation measures – Classical case

classical case: correlation (measure)

- correlation in the state is characterized by $\mathbf{C} = \mathbf{p}_{12} - \mathbf{p}_1 \otimes \mathbf{p}_2$
- let the measure of corr. be the distinguishability of $\mathbf{p}_{12}$ and $\mathbf{p}_1 \otimes \mathbf{p}_2$:

  $$D(\mathbf{p}_{12} || \mathbf{p}_1 \otimes \mathbf{p}_2) = S(\mathbf{p}_1) + S(\mathbf{p}_2) - S(\mathbf{p}_{12}) = I(\mathbf{p}_{12})$$

  this turns out to be the *mutual information* $I(\mathbf{p}_{12})$

a geometric point of view

- it can be proven that: $\operatorname{argmin}_{\mathbf{q}_{12} \in \Delta_{\text{uncorr}}} D(\mathbf{p}_{12} || \mathbf{q}_{12}) = \mathbf{p}_1 \otimes \mathbf{p}_2$, so $\mathbf{p}_1 \otimes \mathbf{p}_2$ is the least distinguishable ("closest") uncorrelated state
- $I(\mathbf{p}_{12})$ can be interpreted as the distinguishability from the least distinguishable uncorrelated state:
  $\min_{\mathbf{q}_{12} \in \Delta_{\text{uncorr}}} D(\mathbf{p}_{12} || \mathbf{q}_{12}) = D(\mathbf{p}_{12} || \mathbf{p}_1 \otimes \mathbf{p}_2) = I(\mathbf{p}_{12})$
- there are other measures of distance in Δ_{12} leading to other measures of correlations, e.g.: $D_\alpha(\mathbf{p}_{12}, \mathbf{p}_1 \otimes \mathbf{p}_2) = \|\mathbf{p}_{12} - \mathbf{p}_1 \otimes \mathbf{p}_2\|_\alpha = \|\mathbf{C}\|_\alpha$

Correlation measures – Quantum case I.: correlation

quantum case I: correlation (measure)

- correlation in the states is characterized by $\Gamma = \varrho_{12} - \varrho_1 \otimes \varrho_2$
- let the measure of corr. be the distinguishability of ϱ_{12} and $\varrho_1 \otimes \varrho_2$

$$D(\varrho_{12}||\varrho_1 \otimes \varrho_2) = S(\varrho_1) + S(\varrho_2) - S(\varrho_{12}) =: I(\varrho_{12})$$

$I(\varrho_{12})$ being the quantum mutual information

quantum mutual information

- vanishes exactly for uncorrelated states
- for pure states, $D(\pi_{12}||\pi_1 \otimes \pi_2) = 2S(\pi_1) = 2S(\pi_2)$

Correlation measures – Quantum case I.: correlation

quantum case I: correlation (measure)

- correlation in the states is characterized by $\Gamma = \varrho_{12} - \varrho_1 \otimes \varrho_2$
- let the measure of corr. be the distinguishability of ϱ_{12} and $\varrho_1 \otimes \varrho_2$

$$D(\varrho_{12}||\varrho_1 \otimes \varrho_2) = S(\varrho_1) + S(\varrho_2) - S(\varrho_{12}) =: I(\varrho_{12})$$

$I(\varrho_{12})$ being the quantum mutual information

a geometric point of view

- again, it can be proven that $\operatorname{argmin}_{\omega_{12}\in\mathcal{D}_{\text{uncorr}}} D(\varrho_{12}||\omega_{12}) = \varrho_1 \otimes \varrho_2$, so $\varrho_1 \otimes \varrho_2$ is the least distinguishable ("closest") uncorrelated state
- $I(\varrho_{12})$ can be interpreted as the distinguishability from the least distinguishable uncorrelated state, $\min_{\omega_{12}\in\mathcal{D}_{\text{uncorr}}} D(\varrho_{12}||\omega_{12}) = D(\varrho_{12}||\varrho_1 \otimes \varrho_2) = I(\varrho_{12})$
- there are other measures of distance in $\mathcal{D}$ leading to other measures of correlations, e.g.: $D_\alpha(\varrho_{12}, \varrho_1 \otimes \varrho_2) = \|\varrho_{12} - \varrho_1 \otimes \varrho_2\|_\alpha = \|\Gamma\|_\alpha$

Correlation measures – Quantum case II.: discord

quantum case II: discord (measure)

- quantum mutual information $I(\varrho_{12}) := S(\varrho_1) + S(\varrho_2) - S(\varrho_{12})$
- what about the other definition, based on conditional state? conditional sate in general is ill-defined in quantum mechanics, however, it can be defined w.r.t. a POVM by $\mathcal{M} = \{\mathcal{M}_i\}$,
- J w.r.t. a POVM: $J_{2|\mathcal{M}}(\varrho_{12}) = S(2) - S_{2|\mathcal{M}}(\varrho_{12})$
  with the cond. entropy (w.r.t. $\mathcal{M}$): $S_{2|\mathcal{M}}(\varrho_{12}) = \sum_i p_i S(\varrho_{2|\mathcal{M}_i})$
  with the cond. state (w.r.t. $\mathcal{M}_i$): $\varrho_{2|\mathcal{M}_i} = \mathrm{Tr}_1(M_i \otimes \mathbf{I})\varrho_{12}(M_i \otimes \mathbf{I})^\dagger$,
- then $J_{2|1}(\varrho_{12}) := \max_{\mathcal{M}} J_{2|\mathcal{M}}(\varrho_{12}) \not\equiv I(\varrho_{12})$
- vanishes exactly for nondiscordant ("classical") states, not symmetric
- discord: $D_{2|1}(\varrho_{12}) = I(\varrho_{12}) - J_{2|1}(\varrho_{12})$, $D_{1|2}(\varrho_{12}) = I(\varrho_{12}) - J_{1|2}(\varrho_{12})$
- for pure states, $D_{2|1}(\pi_{12}) = D_{1|2}(\pi_{12}) = S(\pi_1) = S(\pi_2)$

Correlation measures – Quantum case II.: discord

(information-)geometric measures

- let the *relative entropy of discord* be the distinguishability from the least distinguishable classical state: $\min_{\omega_{12}\in\mathcal{D}_{\text{nondsc}}} D(\varrho_{12}||\omega_{12})$
- there are other measures of distance in $\mathcal{D}$ leading to other measures of discord (*geometric measure of discord*): $\min_{\omega_{12}\in\mathcal{D}_{\text{nondsc}}} \|\varrho_{12}-\omega_{12}\|_\alpha$

Correlation measures – Quantum case III.: entanglement

quantum case III: entanglement (measure)

- quantum case: there are pure states with mixed marginals, so, for pure states, let the measure of entanglement be the mixedness of the subsystem (vanishes exactly for separable pure states)
- *entanglement entropy:* $E(\pi_{12}) = S(\pi_1) = S(\pi_2)$
- for mixed states, *entanglement of formation:*

$$E_{\mathrm{F}}(\varrho_{12}) = \min_{\varrho_{12}=\sum_i p_i \pi_{12;i}} \sum\nolimits_i p_i E(\pi_{12;i})$$

  i.e., "average entanglement entropy of the optimal decomposition"
- vanishes exactly for separable states, $E_{\mathrm{F}}(\pi_{12}) = E(\pi_{12})$ for pure ones
- there are Rényi/Tsallis generalizations, e.g., the concurrence $C = \sqrt{S_2^{\mathrm{Ts}}}$ instead of S leads to the concurrence of formation C_{F}, for two qubits, this is called Wootters concurrence (explicit min!)

Correlation measures – Quantum case III.: entanglement

(information-)geometric measures

- let the *relative entropy of entanglement* be the distinguishability from the least distinguishable separable state: $\min_{\omega_{12}\in\mathcal{D}_{\text{sep}}} D(\varrho_{12}||\omega_{12})$
- there are other measures of distance in $\mathcal{D}$ leading to other measure of ent. (*geom. measure of entanglement*): $\min_{\omega_{12}\in\mathcal{D}_{\text{sep}}} \|\varrho_{12}-\omega_{12}\|_\alpha$

operational measures w.r.t. LQCC protocolls

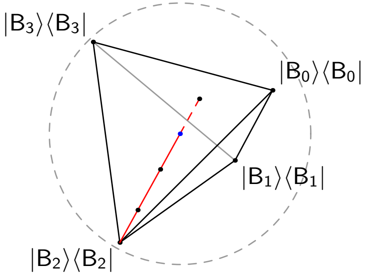
- *distillable entanglement* and *entanglement cost*

$$E_{\text{D}}(\varrho_{12}) = \sup\Big\{ r \;\Big|\; \lim_{m\to\infty}\Big(\inf_{\mathcal{L}\ \text{LQCC}} \big\|\mathcal{L}(\varrho_{12}^{\otimes m}) - (|\text{B}_0\rangle\langle\text{B}_0|)^{\otimes mr}\big\|_1\Big) = 0\Big\}$$

$$E_{\text{C}}(\varrho_{12}) = \inf\Big\{ r \;\Big|\; \lim_{m\to\infty}\Big(\inf_{\mathcal{L}\ \text{LQCC}} \big\|\mathcal{L}\big((|\text{B}_0\rangle\langle\text{B}_0|)^{\otimes mr}\big) - \varrho_{12}^{\otimes m}\big\|_1\Big) = 0\Big\}$$

- for pure states $E_{\text{D}}(\pi_{12}) = E_{\text{C}}(\pi_{12}) = E_{\text{F}}(\pi_{12}) = E(\pi_{12}) = S(\pi_1)$
- there are undistillable states (bound entangled) $\mathcal{D}_{\text{sep}} \subset \mathcal{D}_{\text{bound}} \subset \mathcal{D}_{12}$

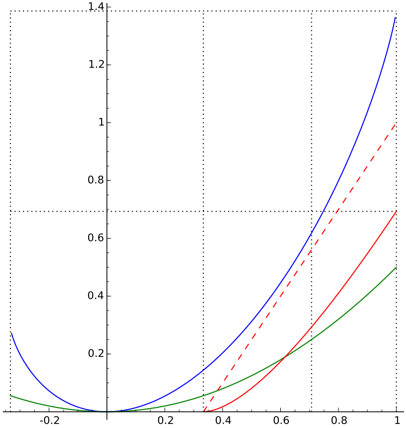
Measures of quantum correlations



examples: Werner states ($d_1 = d_2 = 2$)

- $\varrho_{12} = w|B_2\rangle\langle B_2| + (1-w)\frac{1}{4}\mathbf{I}\otimes\mathbf{I}$ for $-1/3 \leq w \leq 1$
- uncorrelated, classical: $w = 0$, separable $w \leq 1/3$, LHVM for CHSH: $w \leq 1/\sqrt{2}$

Measures of quantum correlations



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- $\varrho_{12} = w|B_2\rangle\langle B_2| + (1-w)\frac{1}{4}\mathbf{I}\otimes\mathbf{I}$ for $-1/3 \leq w \leq 1$
- uncorrelated, classical: $w = 0$, separable $w \leq 1/3$, LHVM for CHSH: $w \leq 1/\sqrt{2}$
- correlation (blue): $I(\varrho_{12}) = 2\ln 2 - S(\varrho_{12})$
- geom. discord (green): $\min_{\omega_{12}\in\mathcal{D}_{\text{nondsc}}} \|\varrho_{12} - \omega_{12}\|^2 = w^2/2$
- Wootters concurrence (dashed red): $C_F(\varrho_{12}) = (3w-1)/2$, $(1/3 \leq w)$
- entanglement of formation (red): $E_F(\varrho_{12})$ through $C_F(\varrho_{12})$

Measures of quantum correlations – w.r.t. quantum maps

creation

- all *uncorrelated* states can be created by *LC* from pure product state (assuming that LC is w.r.t. the local pure states)
- all *nondisc.* states can be created by *LC+CC* from pure product state (or from uncorrelated state) (assuming that LC is w.r.t. the local pure states)
- all *separable* states can be created by *LQ+CC* from pure product state

monotonity

- *correlation:* quantity/notion which doesn't increase under *LC* (it can increase if CC is allowed) (works only for nondiscordant states)
- *discord:* quantity/notion which doesn't increase under *LC+CC* (but it can increase if LQ is allowed) doesn't make sense!
- *entanglement:* quantity/notion which doesn't incr. under *LQ+CC* (it can increase only if QC is allowed) (distant lab paradigm)

Criteria of correlations – Overview

in general

- task: decide whether a state shows correlation/discord/entanglement (we are usually not able to evaluate a discord/entanglement measure)
- deciding whether a classical state $\mathbf{p}_{12} \in \Delta_{12}$ is *uncorrelated* is easy:

$$\mathbf{p}_{12} \in \Delta_{\text{uncorr}} \qquad \Longleftrightarrow \qquad \mathbf{p}_{12} = (\text{Sum}_2\, \mathbf{p}_{12}) \otimes (\text{Sum}_1\, \mathbf{p}_{12})$$

- deciding whether a quantum state $\varrho_{12} \in \mathcal{D}_{12}$ is *uncorrelated* is easy:

$$\varrho_{12} \in \mathcal{D}_{\text{uncorr}} \qquad \Longleftrightarrow \qquad \varrho_{12} = (\text{Tr}_2\, \varrho_{12}) \otimes (\text{Tr}_1\, \varrho_{12})$$

- deciding whether a quantum state $\varrho_{12} \in \mathcal{D}_{12}$ is *nondiscordant* is not so simple, but there exists a necessary and sufficient criterion:

$$\varrho_{12} \in \mathcal{D}_{\text{nondsc}} \qquad \Longleftrightarrow \qquad \text{a condition is fulfilled}$$

- deciding whether a quantum state $\varrho_{12} \in \mathcal{D}_{12}$ is *separable* is a hard optimization task, however, there are several necessary but not sufficient criteria, easy to check (and also interesting):

$$\varrho_{12} \in \mathcal{D}_{\text{sep}} \qquad \Longrightarrow \qquad \text{a condition is fulfilled}$$

Criteria of correlations – Quantum case III.: entanglement

criteria by majorization

- separable states:
  "the whole system is more disordered than any of its subsystems"

$$\varrho_{12} \in \mathcal{D}_{\text{sep}} \quad \Longrightarrow \quad \varrho_{12} \preceq \varrho_1 \quad \text{and} \quad \varrho_{12} \preceq \varrho_2$$

criteria by entropies

- entropic reformulation of the above:

$$\varrho_{12} \in \mathcal{D}_{\text{sep}} \quad \Longrightarrow \quad S(\varrho_{12}) \geq S(\varrho_1) \quad \text{and} \quad S(\varrho_{12}) \geq S(\varrho_2)$$

  e.g.: von Neumann entropy (Rényi, Tsallis are also suitable)
- specially for $\pi_{12} = |\psi_{12}\rangle\langle\psi_{12}| \in \mathcal{P}_{12}$ pure state: $S(\pi_{12}) = 0$

$$\pi_{12} \in \mathcal{P}_{\text{sep}} \quad \Longleftrightarrow \quad S(\pi_1) = 0 \quad \text{and} \quad S(\pi_2) = 0$$

(as we have already seen)

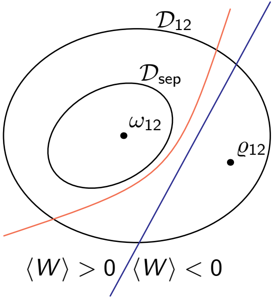
Criteria of correlations – Quantum case III.: entanglement

criteria by Clauser-Horne-Shimony-Holt (CHSH) inequalities (Bell-variant)

- spin-correlation experiment (CHSH-setting), with observable $B_{\mathbf{a},\mathbf{a}',\mathbf{b},\mathbf{b}'} = \mathbf{a}\boldsymbol{\sigma}\otimes\mathbf{b}\boldsymbol{\sigma} + \mathbf{a}\boldsymbol{\sigma}\otimes\mathbf{b}'\boldsymbol{\sigma} + \mathbf{a}'\boldsymbol{\sigma}\otimes\mathbf{b}\boldsymbol{\sigma} - \mathbf{a}'\boldsymbol{\sigma}\otimes\mathbf{b}'\boldsymbol{\sigma}$
- CHSH inequality for local hidden variable model (LHVM):
$$|\mathrm{Tr}(\varrho_{12}B_{\mathbf{a},\mathbf{a}',\mathbf{b},\mathbf{b}'})| \leq 2 \quad \text{for all settings} \qquad \Longleftarrow \qquad \text{LHVM}$$
- for pure states:
$$\varrho_{12} \in \mathcal{P}_{\text{sep}} \qquad \Longleftrightarrow \qquad |\mathrm{Tr}(\varrho_{12}B_{\mathbf{a},\mathbf{a}',\mathbf{b},\mathbf{b}'})| \leq 2 \quad \text{for all settings}$$
- usually not enough for mixed states:
$$\varrho_{12} \in \mathcal{D}_{\text{sep}} \qquad \Longrightarrow \qquad |\mathrm{Tr}(\varrho_{12}B_{\mathbf{a},\mathbf{a}',\mathbf{b},\mathbf{b}'})| \leq 2 \quad \text{for all settings}$$

  there are entangled states admitting LHVM for CHSH (Werner)

Criteria of correlations – Quantum case III.: entanglement



criteria by witnesses

- "entanglement witness": $W \in \operatorname{Lin}\mathcal{H}$ observable, $\forall \omega_{12} \in \mathcal{D}_{\text{sep}} : \operatorname{Tr} W\omega_{12} \geq 0$ and $\exists \varrho_{12} \in \mathcal{D}_{12} \setminus \mathcal{D}_{\text{sep}} : \operatorname{Tr} W\varrho_{12} < 0$
- witnesses can be found for all entangled states
- "clipping around the convex set $\mathcal{D}_{\text{sep}}$"
- $W_{\text{CHSH}} = 2\mathbf{I}\otimes\mathbf{I} - B_{\mathbf{a},\mathbf{a}',\mathbf{b},\mathbf{b}'}$ "CHSH-witness" (not sufficient)
- there are also nonlinear criteria, e.g., nonlinear Bell-inequalities.

$$\varrho_{12} \in \mathcal{D}_{\text{sep}} \quad\Longleftrightarrow\quad \langle W\rangle \equiv \operatorname{Tr} W\varrho_{12} \geq 0 \quad \text{for all withesses } W$$

$$\varrho_{12} \in \mathcal{D}_{\text{sep}} \quad\Longrightarrow\quad \langle W\rangle \equiv \operatorname{Tr} W\varrho_{12} \geq 0 \quad \text{for some withesses } W$$

Criteria of correlations – Quantum case III.: entanglement

criteria by positive maps

- physics: completely positive maps $\mathcal{E} : \mathrm{Lin}\,\mathcal{H}_1 \to \mathrm{Lin}\,\mathcal{H}_1$
  preserve the positivity of not only the system ($\mathcal{E}(\varrho_1) \geq 0$),
  but also of the sys. together with its environment ($(\mathcal{E} \otimes \mathcal{I})(\varrho_{12}) \geq 0$)
- positive but not completely positive maps: $\mathcal{F} : \mathrm{Lin}\,\mathcal{H}_1 \to \mathrm{Lin}\,\mathcal{H}_1$

$$\varrho_{12} \in \mathcal{D}_{\text{sep}} \quad \Longleftrightarrow \quad \big(\mathcal{F} \otimes \mathcal{I}\big)\big(\varrho_{12}\big) \geq 0 \quad \text{for all pos. maps } \mathcal{F}$$

$$\varrho_{12} \in \mathcal{D}_{\text{sep}} \quad \Longrightarrow \quad \big(\mathcal{F} \otimes \mathcal{I}\big)\big(\varrho_{12}\big) \geq 0 \quad \text{for some pos. maps } \mathcal{F}$$

examples

- *positive partial transpose* criterion (Peres): $\mathcal{F}(\omega) = \omega^{\mathrm{T}}$
- *reduction* criterion (Horodecki): $\mathcal{F}(\omega) = (\mathrm{Tr}\,\omega)\mathbf{I} - \omega$
- many others...

Outline

References

. . . Some of these can be found in a secret suitcase ;-)

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  Kavan Modi, *A Pedagogical Overview of Quantum Discord,*
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  The classical-quantum boundary for correlations: Discord and related measures,
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- A really pleasurable book on quantum states:

  Ingemar Bengtsson and Karol Życzkowski, *Geometry of Quantum States: An Introduction to Quantum Entanglement,*
  Cambridge University Press, 2006

- Some topics on the entanglement of multipartite mixed states (of personal taste)

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Statement:

Thank you for your attention!

Corollary:

(: Have a nice weekend! :)