Entanglement depths and metrological entanglement criteria[1]

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Permutation invariant multipartite entanglement properties [1, 2, 4, 5]

- state vector: $|\psi\rangle \in \mathcal{H}$ normalized
- pure state: $\pi = |\psi\rangle\langle\psi| \in \mathcal{P} = \mathrm{Extr}(\mathcal{D})$
- (mixed) state: $\rho = \sum_j p_j \pi_j \in \mathcal{D} = \mathrm{Conv}(\mathcal{P})$
- system: $L = \{1, 2, \dots, n\}$, subsystem: $X \subseteq L$, then $\mathcal{H}_X$, $\mathcal{P}_X$, $\mathcal{D}_X$

ξ-separability

- separability w.r.t. a (set) partition: $\xi = \{X_1, X_2, \dots\}$
- refinement: $\upsilon \preceq \xi$ def.: $\forall Y \in \upsilon, \exists X \in \xi : Y \subseteq X$
- ξ-separable states: $\mathcal{D}_\xi = \mathrm{Conv}\{\bigotimes_{X\in\xi} \rho_X\}$
  LOCC-closed $\qquad \upsilon \preceq \xi \iff \mathcal{D}_\upsilon \subseteq \mathcal{D}_\xi$

$\hat{\xi}$-separability

- separability w.r.t. an integer partition $\qquad$ 'type of ξ' (Young diagram)
  $\hat{\xi} = \{x_1, x_2, \dots\} = \{|X_1|, |X_2|, \dots\} \longleftarrow \xi = \{X_1, X_2, \dots\}$
- refinement: $\hat{\upsilon} \preceq \hat{\xi}$ def.: $\exists \upsilon \preceq \xi$ of those types
- $\hat{\xi}$-separable states: $\mathcal{D}_{\hat{\xi}} = \mathrm{Conv} \bigcup_{\xi \mapsto \hat{\xi}} \{\bigotimes_{X\in\xi} \rho_X\}$
  LOCC-closed $\qquad \hat{\upsilon} \preceq \hat{\xi} \iff \mathcal{D}_{\hat{\upsilon}} \subseteq \mathcal{D}_{\hat{\xi}}$

permutation invariant properties

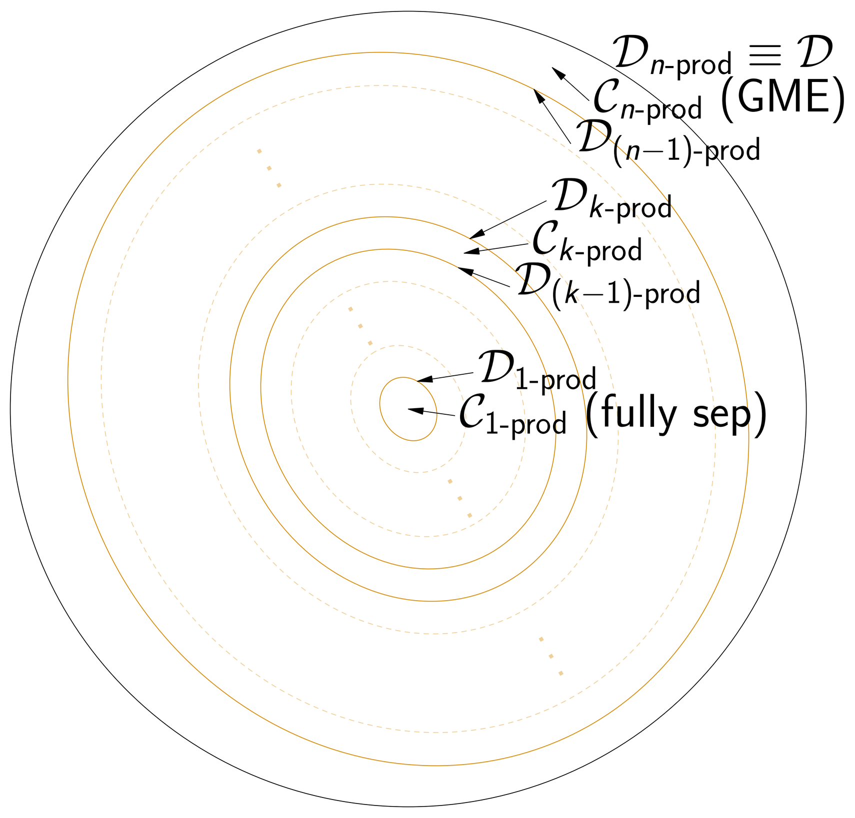
- separability w.r.t. multiple integer partitions $\hat{\boldsymbol{\xi}} = \{\hat{\xi}_1, \hat{\xi}_2, \dots\}$
- refinement: $\hat{\boldsymbol{\upsilon}} \preceq \hat{\boldsymbol{\xi}}$ def.: $\hat{\upsilon} \subseteq \hat{\xi}$
- $\hat{\boldsymbol{\xi}}$-separable states: $\mathcal{D}_{\hat{\boldsymbol{\xi}}} = \mathrm{Conv} \bigcup_{\hat{\xi}\in\hat{\boldsymbol{\xi}}} \bigcup_{\xi \mapsto \hat{\xi}} \{\bigotimes_{X\in\xi} \rho_X\}$
  LOCC-closed $\qquad \hat{\boldsymbol{\upsilon}} \preceq \hat{\boldsymbol{\xi}} \iff \mathcal{D}_{\hat{\boldsymbol{\upsilon}}} \subseteq \mathcal{D}_{\hat{\boldsymbol{\xi}}}$

Examples: partitionability, producibility and stretchability [1, 2]

- height, width and rank of Young diagrams

$$h(\hat{\xi}) = |\hat{\xi}| \qquad \hat{\xi}_{k\text{-part}} = \{\hat{\xi} \mid h(\hat{\xi}) \geq k\} \qquad \hat{\xi}_{k\text{-part}} \preceq \hat{\xi}_{k'\text{-part}} \iff k \geq k'$$
$$w(\hat{\xi}) = \max \hat{\xi} \qquad \hat{\xi}_{k\text{-prod}} = \{\hat{\xi} \mid w(\hat{\xi}) \leq k\} \qquad \hat{\xi}_{k\text{-part}} \preceq \hat{\xi}_{k'\text{-prod}} \iff k \leq k'$$
$$r(\hat{\xi}) = w(\hat{\xi}) - h(\hat{\xi}) \qquad \hat{\xi}_{k\text{-str}} = \{\hat{\xi} \mid r(\hat{\xi}) \leq k\} \qquad \hat{\xi}_{k\text{-str}} \preceq \hat{\xi}_{k'\text{-str}} \iff k \leq k'$$

- monotones: $\hat{\upsilon} \prec \hat{\xi} \;\Rightarrow\; h(\hat{\upsilon}) > h(\hat{\xi})$, $w(\hat{\upsilon}) \leq w(\hat{\xi})$, $r(\hat{\upsilon}) < r(\hat{\xi})$
- k-prod. states: $\mathcal{D}_{k\text{-prod}} = \mathcal{D}_{\hat{\xi}_{k\text{-prod}}}$ (LOCC-closed, nested)
  strictly k-prod. states: $\mathcal{C}_{k\text{-prod}} = \mathcal{D}_{k\text{-prod}} \setminus \mathcal{D}_{(k-1)\text{-prod}}$ class (disjoint)



- depth of partitionability, producibility, and stretchability:

$$D_{\text{part}}(\rho) := \max\{k \mid \rho \in \mathcal{D}_{k\text{-part}}\}$$
$$D_{\text{prod}}(\rho) := \min\{k \mid \rho \in \mathcal{D}_{k\text{-prod}}\} \equiv D(\rho)$$
$$D_{\text{str}}(\rho) := \min\{k \mid \rho \in \mathcal{D}_{k\text{-str}}\}$$

- LOCC-monotone (decreasing) and $\mathcal{C}_{k\text{-prod}} = D^{-1}_{\text{prod}}(k)$

One parameter properties, depths [1]

- generator function: any $\hat{\xi} \mapsto f(\hat{\xi})$ increasing or decreasing monotone
  $$\hat{\upsilon} \preceq \hat{\xi} \implies f(\hat{\upsilon}) \lesseqgtr f(\hat{\xi})$$
- one-parameter properties: for the values k of f down-sets of integer partitions:
  $\hat{\xi}_{k,f} = \{\hat{\xi} \mid f(\hat{\xi}) \lesseqgtr k\}$ $\qquad$ chain: $k \lesseqgtr k' \iff \hat{\xi}_{k,f} \preceq \hat{\xi}_{k',f}$
- spaces of (k,f)-separable states:
  $\mathcal{D}_{k,f} = \mathcal{D}_{\hat{\xi}_{k,f}}$ $\qquad$ LOCC-closed, nested: $k \lesseqgtr k' \iff \mathcal{D}_{k,f} \subseteq \mathcal{D}_{k',f}$
- classes of strictly (k,f)-separable states:
  $\mathcal{C}_{k,f} = \mathcal{D}_{k,f} \setminus \mathcal{D}_{k_\mp,f}$ $\qquad$ disjoint, LOCC convertibility: $k \lesseqgtr k' \implies \mathcal{C}_{k,f} \leftarrow \mathcal{C}_{k',f}$
  (there exists state and LOCC...)
- beyond the usual, relative entropy based entanglement measures [2, 4, 5], we have f-entanglement depth (discrete):
  $$D_f(\rho) = \begin{cases} \min\{k \mid \rho \in \mathcal{D}_{k,f}\} & \text{if } f \text{ is increasing} \\ \max\{k \mid \rho \in \mathcal{D}_{k,f}\} & \text{if } f \text{ is decreasing} \end{cases}$$
- LOCC-monotone (decreasing/increasing) and $\mathcal{C}_{k,f} = D_f^{-1}(k)$
- further example: squareability, avg
  $$s_2(\hat{\xi}) = \sum_{x\in\hat{\xi}} x^2 = n \sum_{x\in\hat{\xi}} \frac{x}{n} x = n \,\mathrm{avg}(\hat{\xi})$$
  average size of parts/subsystems (w.r.t. picking elementary subsystem)

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Illustration: partitionability and producibility for $n = 2, 3, 4$ [4, 5]

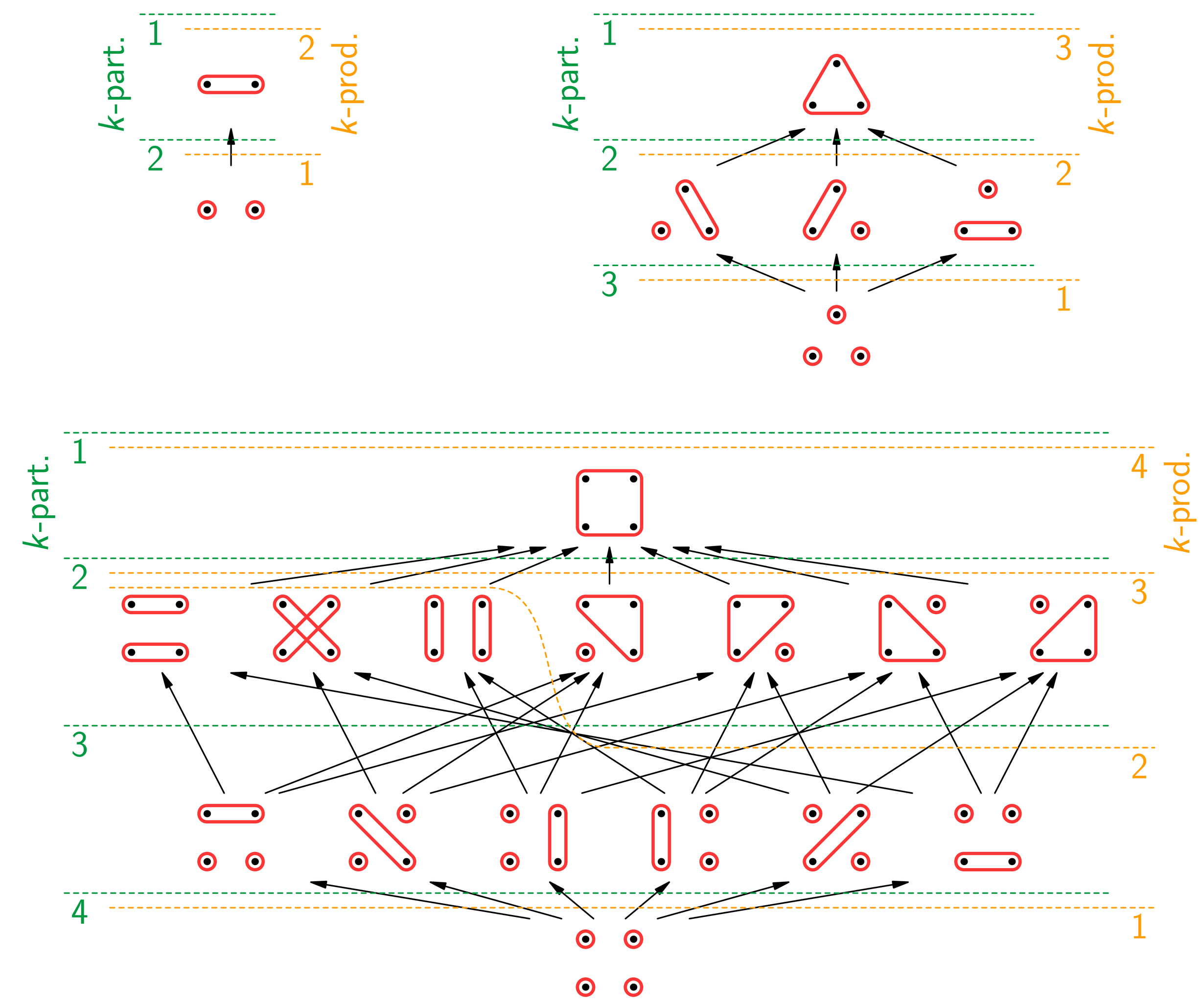
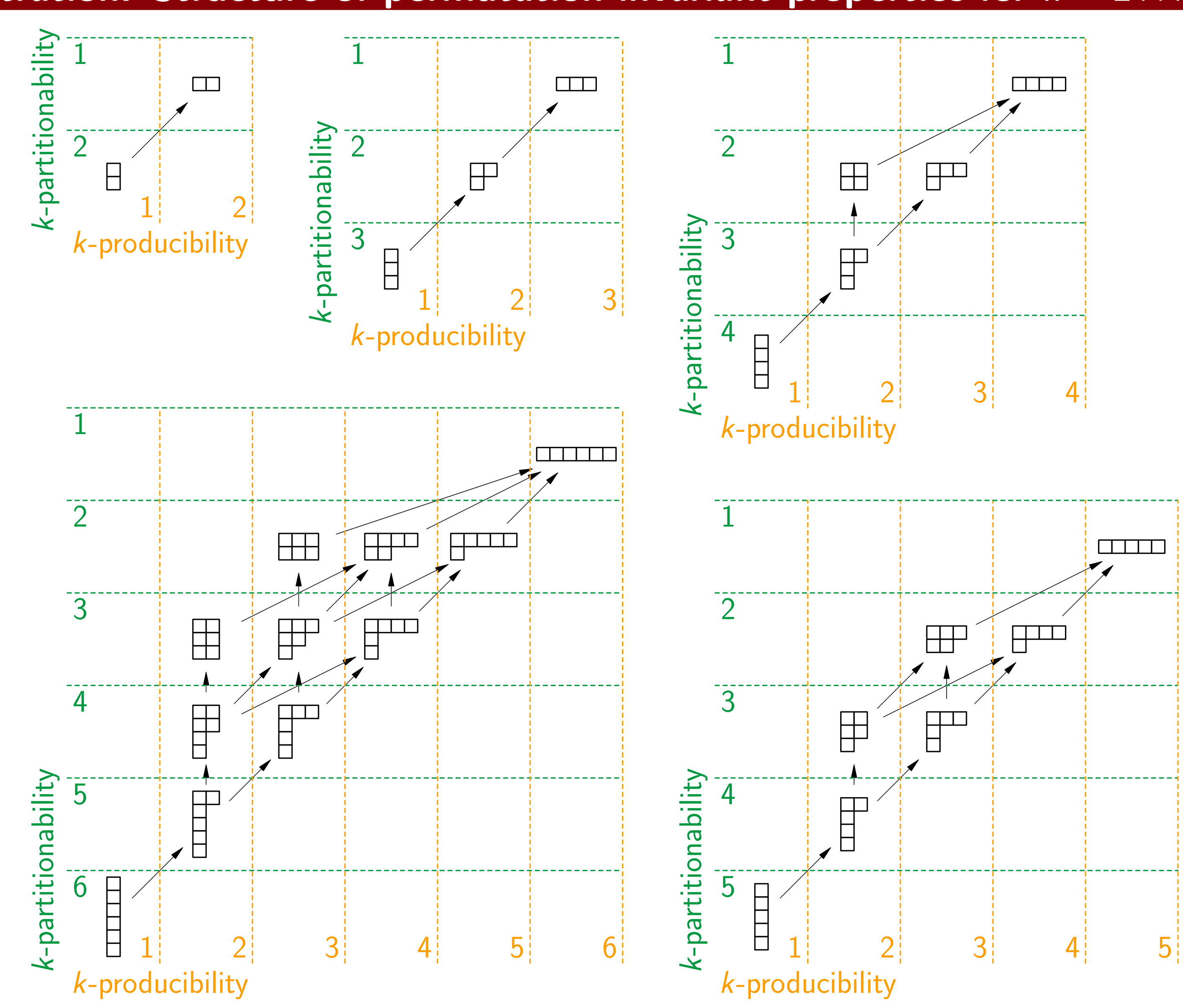


Illustration: Structure of permutation invariant properties for $n = 2 \dots 6$ [2]



Depth of Formation [1]

- f-entanglement depth: D_f, discrete measure
- $\rho_\epsilon := \epsilon|\psi_{\text{GHZ}}\rangle\langle\psi_{\text{GHZ}}| + (1-\epsilon)|\psi_{\text{sep}}\rangle\langle\psi_{\text{sep}}|$ $\qquad$ with $\langle\psi_{\text{sep}}|\psi_{\text{GHZ}}\rangle = 0$
  problem: entanglement depth $D(\rho_\epsilon) = n$ for all $\epsilon > 0$
- f-entanglement depth of formation: convex/concave roof extension (continuous meas.)
  $$D_f^{\text{oF}}(\rho) := \begin{cases} \min\limits_{\{(p_j,\pi_j)\}\vdash\rho} \sum_j p_j D_f(\pi_j) & \text{if } f \text{ is increasing} \\ \max\limits_{\{(p_j,\pi_j)\}\vdash\rho} \sum_j p_j D_f(\pi_j) & \text{if } f \text{ is decreasing} \end{cases}$$
- problem solved: $1 < D^{\text{oF}}(\rho_\epsilon) \leq \epsilon n + (1-\epsilon)1$ if $\epsilon > 0$
- note also that
  $$D_f(\rho) := \begin{cases} \min\limits_{\{(p_j,\pi_j)\}\vdash\rho} \max\limits_j D_f(\pi_j) & \text{if } f \text{ is increasing} \\ \max\limits_{\{(p_j,\pi_j)\}\vdash\rho} \min\limits_j D_f(\pi_j) & \text{if } f \text{ is decreasing} \end{cases}$$

Metrological entanglement criteria [1]

- estimation of parameter θ in the dynamics $U(\theta) = e^{-iA\theta}$
- Cramér-Rao bound $(\Delta\theta)^2 \geq \frac{1}{NF_{\text{Q}}(\rho,A)}$, on the precision by quantum Fisher information
- separability criteria
  $$\begin{array}{ccccc} & & D^{\text{oF}}(\rho) & \leq & D(\rho) \quad \text{(prod.)} \\ & & \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} \\ F_{\text{Q}}(\rho, J^z)/n & \leq & D^{\text{oF}}_{\text{avg}}(\rho) & \leq & D_{\text{avg}}(\rho) \quad \text{(avg.)} \end{array}$$
- (prod-)entanglement depth $\qquad D(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \max_j D(\pi_j)$
- (prod-)entanglement depth of formation $\qquad D^{\text{oF}}(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \sum_j p_j D(\pi_j)$
- avg-entanglement depth $\qquad D_{\text{avg}}(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \max_j D_{\text{avg}}(\pi_j)$
- avg-entanglement depth of formation $\qquad D^{\text{oF}}_{\text{avg}}(\rho) = \min_{\{(p_j,\pi_j)\}\vdash\rho} \sum_j p_j D_{\text{avg}}(\pi_j)$

average size of entangled subsystems (ASES)

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