Partial separability revisited

Szilárd Szalay[1,2] and Zoltán Kökényesi[2]

[1]Strongly Correlated Systems "Lendület" Research Group – Wigner Research Centre for Physics

[2]Department of Theoretical Physics – Budapest University of Technology and Economics

1. The basic questions of entanglement theory

- the structure of entanglement (interesting for more-than-two subsystems)
- the qualification of entanglement (separability criteria)
- the quantification of entanglement (entanglement measures)

These are difficult for mixed states. Here we present some sort of answers for these questions. We recall the *bipartite case*, we show the *tripartite case* in details, and we refer to [3] for the *n-partite case*.

2. Setting the stage

- state vector: $|\psi\rangle \in \mathcal{H}$ (normalized)
- pure state: $\pi = |\psi\rangle\langle\psi| \in \mathcal{P}$
- mixed state (of an ensemble): $\varrho = \sum_j p_j \pi_j \in \mathcal{D} = \operatorname{Conv}\mathcal{P}$
- mixedness: e.g. von Neumann entropy $S(\varrho) = -\operatorname{tr}\varrho\ln\varrho$

3. Bipartite systems and entanglement

Pure states

- $|\psi\rangle \in \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \quad \rightsquigarrow \quad |\psi\rangle\langle\psi| = \pi \in \mathcal{P}$
- separable: $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \quad \rightsquigarrow \quad \pi = \pi_1 \otimes \pi_2 \in \mathcal{P}_{\text{sep}} \subset \mathcal{P}$
- else it is entangled ($\mathcal{P} \setminus \mathcal{P}_{\text{sep}}$). Then measurement on subsystem 1 causes the collapse of the state of subsystem 2. (worry of EPR)
- state of subsystems (e.g. $\pi_1 = \operatorname{tr}_2 \pi \in \mathcal{D}_1$) are not necessary pure
- ($|\psi\rangle$) π is entangled if (and only if) π_1 and π_2 are mixed. In this case, *"the best possible knowledge of the whole does not involve the best possible knowledge of its parts."* (Schrödinger)
- the *mixedness of the subsystems* $E(\pi) = S(\pi_1)$ is a good measure of entanglement (entanglement monotone)
- vanishes exactly for separable states ("indicator function")

Mixed states

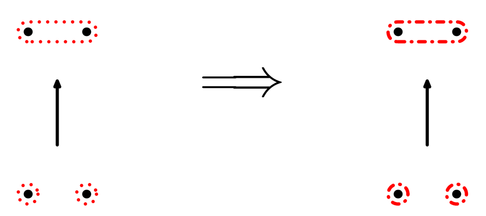
- a mixed state is separable if there exists separable decomposition: $\varrho = \sum_i p_i \pi_{i,1} \otimes \pi_{i,2} \in \mathcal{D}_{\text{sep}} \subset \mathcal{D}$
- motivation: classically correlated sources produce states of this kind (Werner)
- else it is entangled ($\mathcal{D} \setminus \mathcal{D}_{\text{sep}}$)
- the decomposition is not unique, decision of its existence is difficult
- the *average entanglement of the optimal decomposition* (convex roof extension)

$$E(\pi) = S(\pi_1) \qquad E : \mathcal{P} \to [0, S_{\max}]$$

$$\rightsquigarrow \quad E^{\cup}(\varrho) = \min_{\varrho=\sum_i p_i\pi_i} \sum_i p_i E(\pi_i) \qquad E^{\cup} : \mathcal{D} \to [0, S_{\max}]$$

is a good measure of entanglement (entanglement monotone)
- vanishes exactly for separable states ("indicator function")

4. Bipartite systems: A more abstract approach



Class	$\mathcal{D}_{\text{sep}}$	$\mathcal{D}$
$\mathcal{C}_{\text{sep}}$	$\subset$	$\subset$
$\mathcal{C}_{\text{ent}}$	$\not\subset$	$\subset$

- Pure states: (closed, containing) $\mathcal{P}_{\text{sep}} \subset \mathcal{P}$
- Mixed states: (convex hulls, closed, containing) $\mathcal{D}_{\text{sep}} = \operatorname{Conv}\mathcal{P}_{\text{sep}} \subset \mathcal{D} = \operatorname{Conv}\mathcal{P}$
- Classes: (intersections of these, disjoint)
  - $\mathcal{C}_{\text{sep}} = \mathcal{D}_{\text{sep}}$ separable
  - $\mathcal{C}_{\text{ent}} = \mathcal{D} \setminus \mathcal{D}_{\text{sep}}$ entangled
- Indicator functions:
  - $\pi \in \mathcal{P}_{\text{sep}} \iff E(\pi) = 0$
  - $\varrho \in \mathcal{D}_{\text{sep}} \iff E^{\cup}(\varrho) = 0$

7. Summary, questions

- classification: considering the partial separability properties in the fullest detail
- countable many classes, for arbitrary number of subsystems [3]
- independent of the dimensions of the Hilbert spaces of the subsystems
- structure of classification is complicated, but classes can be merged
- q: How to utilize these different kinds of entanglement?
- qualification: necessary and sufficient criteria
- hard optimization task (other criteria suffer from this as well)
- ind. functions have a transparent structure, reflecting that of the classification
- quantification: via entanglement monotone indicator functions
- q: What is the meaning of means of entropies of *different* subsystems?

8. Acknowledgement

This project was supported by the New Hungary Development Plan (project ID: TÁMOP-4.2.1.B-09/1/KMR-2010-2), the New Széchenyi Plan of Hungary (project ID: TÁMOP-4.2.2.B-10/1–2010-9) and the "Lendület" program of the Hungarian Academy of Sciences (project ID: 81010-00).

5. Tripartite systems: Partial separability

Pure states

- different kinds of separabilities
- notation: for all distinct $a, b, c \in \{1, 2, 3\}$

$$\mathcal{P}_{123} \equiv \mathcal{P}$$

$$\mathcal{P}_{a|bc} = \{\pi_a \otimes \pi_{bc}\}$$

$$\mathcal{P}_{1|2|3} = \{\pi_1 \otimes \pi_2 \otimes \pi_3\}$$

- closed, containing

Mixed states

- partial separability *in the fullest detail:* convex hulls of all possible closed sets

$$\mathcal{D}_{123} = \operatorname{Conv}(\mathcal{P}_{123})$$

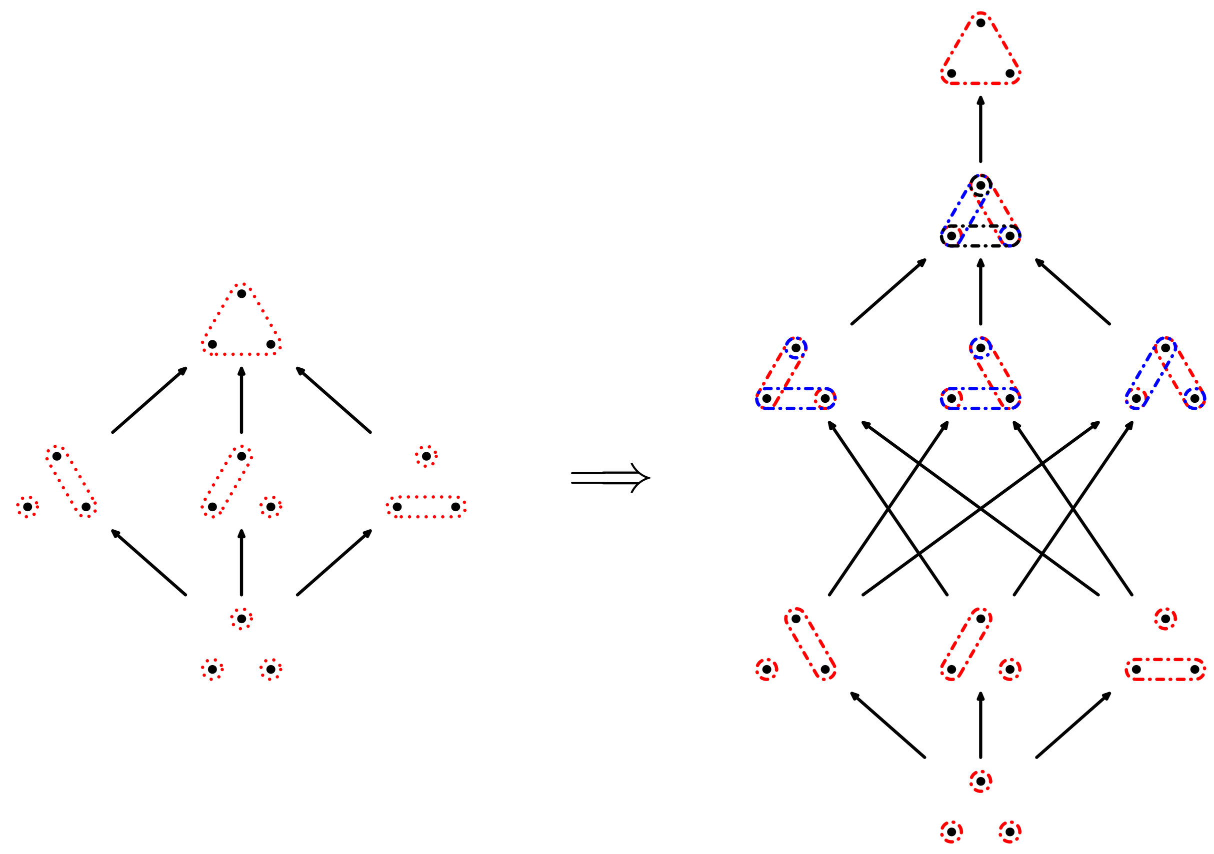
$$\mathcal{D}_{1|23,2|13,3|12} = \operatorname{Conv}(\mathcal{P}_{1|23} \cup \mathcal{P}_{2|13} \cup \mathcal{P}_{3|12})$$

$$\mathcal{D}_{b|ac,c|ab} = \operatorname{Conv}(\mathcal{P}_{b|ac} \cup \mathcal{P}_{c|ab})$$

$$\mathcal{D}_{a|bc} = \operatorname{Conv}(\mathcal{P}_{a|bc})$$

$$\mathcal{D}_{1|2|3} = \operatorname{Conv}(\mathcal{P}_{1|2|3})$$

- closed, containing



- classes arise as the intersections of these (see [3] for interesting details)

class	$\mathcal{D}_{1\|2\|3}$	$\mathcal{D}_{a\|bc}$	$\mathcal{D}_{b\|ac}$	$\mathcal{D}_{c\|ab}$	$\mathcal{D}_{b\|ac,c\|ab}$	$\mathcal{D}_{a\|bc,c\|ab}$	$\mathcal{D}_{a\|bc,b\|ac}$	$\mathcal{D}_{1\|23,2\|13,3\|12}$	$\mathcal{D}_{123}$	class in [2]	class in [1]
$\mathcal{C}_3$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	3	5
$\mathcal{C}_{2.8}$	$\not\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	2.8	4
$\mathcal{C}_{2.7.a}$	$\not\subset$	$\not\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	2.7, 2.6, 2.5	3.3, 3.2, 3.1
$\mathcal{C}_{2.6.a}$	$\not\subset$	$\subset$	$\not\subset$	$\not\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	2.4, 2.3, 2.2	2.3, 2.2, 2.1
$\mathcal{C}_{2.5.a}$	$\not\subset$	$\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\subset$	$\subset$	$\subset$	$\subset$	2.4, 2.3, 2.2	2.3, 2.2, 2.1
$\mathcal{C}_{2.4}$	$\not\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\subset$	$\subset$	$\subset$	$\subset$	$\subset$	2.1	1
$\mathcal{C}_{2.3.a}$	$\not\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\subset$	$\subset$	$\subset$	$\subset$	2.1	1
$\mathcal{C}_{2.2.a}$	$\not\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\subset$	$\not\subset$	$\not\subset$	$\subset$	$\subset$	2.1	1
$\mathcal{C}_{2.1}$	$\not\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\subset$	$\subset$	2.1	1
$\mathcal{C}_1$	$\not\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\not\subset$	$\subset$	1	1

6. Tripartite systems: Indicator functions

- pure states: constructed e.g. from entropies of subsystems

$$f_{1|2|3}(\pi) = f_{1|23}(\pi) + f_{2|13}(\pi) + f_{3|12}(\pi)$$

$$f_{a|bc}(\pi) = S(\pi_a)$$

$$f_{b|ac,c|ab}(\pi) = f_{b|ac}(\pi) f_{c|ab}(\pi)$$

$$f_{1|23,2|13,3|12}(\pi) = f_{1|23}(\pi) f_{2|13}(\pi) f_{3|12}(\pi)$$

- indicator functions: appropriate vanishing properties (necessary and sufficient)
  - $\pi \in \mathcal{P}_{1|2|3} \iff f_{1|2|3}(\pi) = 0$
  - $\pi \in \mathcal{P}_{a|bc} \iff f_{a|bc}(\pi) = 0$
  - $\pi \in \mathcal{P}_{b|ca} \cup \mathcal{P}_{c|ab} \iff f_{b|ac,c|ab}(\pi) = 0$
  - $\pi \in \mathcal{P}_{1|23} \cup \mathcal{P}_{2|13} \cup \mathcal{P}_{3|12} \iff f_{1|23,2|13,3|12}(\pi) = 0$
- mixed states: convex roof extensions of pure state indicator functions
  - $\varrho \in \mathcal{D}_{1|2|3} \iff f^{\cup}_{1|2|3}(\varrho) = 0$
  - $\varrho \in \mathcal{D}_{a|bc} \iff f^{\cup}_{a|bc}(\varrho) = 0$
  - $\varrho \in \mathcal{D}_{b|ac,c|ab} \iff f^{\cup}_{b|ac,c|ab}(\varrho) = 0$
  - $\varrho \in \mathcal{D}_{1|23,2|13,3|12} \iff f^{\cup}_{1|23,2|13,3|12}(\varrho) = 0$
- geometric means instead of products lead to entanglement monotones [3]
- three qubits [4]: a different set of ind. functions [3] from SLOCC covariants [5]

9. Bibliography

[1] W. Dür, J. I. Cirac, Classification of multiqubit mixed states: Separability and distillability properties, PRA**61**, 042314 (2000).

[2] M. Seevinck, J. Uffink, Partial separability and entanglement criteria for multiqubit quantum states, PRA**78**, 032101 (2008).

[3] Sz. Szalay, Z. Kökényesi, Partial separability revisited: Necessary and sufficient criteria, PRA**86**, 032341 (2012).

[4] A. Acín, D. Bruß, M. Lewenstein, A. Sanpera, Classification of Mixed Three-Qubit States, PRL**87**, 040401 (2001).

[5] L. Borsten, D. Dahanayake, M. J. Duff, W. Rubens, H. Ebrahim, Freudenthal triple classification of three-qubit entanglement, PRA**80**, 032326 (2009).

Mail: szalay@phy.bme.hu, kokenyesiz@gmail.com