k-partitionability, k-producibility and k-stretchability of entanglement

Szilárd Szalay

Strongly Correlated Systems "Lendület" Research Group – Wigner Research Centre for Physics, Budapest, Hungary

Structure of properties [2, 3, 4]

- state vector: $|\psi\rangle \in \mathcal{H}$ normalized
- pure state: $\pi = |\psi\rangle\langle\psi| \in \mathcal{P} = \operatorname{Extr}\mathcal{D}$
- (mixed) state: $\varrho = \sum_j p_j \pi_j \in \mathcal{D} = \operatorname{Conv}\mathcal{P}$

Level 0: subsystems

- system: $L = \{1, 2, \dots, n\}$
- subsystem: $X \subseteq L$, then $\mathcal{H}_X$, $\mathcal{P}_X$, $\mathcal{D}_X$ — Boolean lattice structure: $P_0 = 2^L$

Level I: set partitions [3, 4, 5, 6, 7]

- partitions of the system: $\xi = \{X_1, X_2, \dots, X_{|\xi|}\} \equiv X_1|X_2|\dots|X_{|\xi|} \in \Pi(L)$
- refinement: $\upsilon \preceq \xi$ def.: $\forall Y \in \upsilon, \exists X \in \xi : Y \subseteq X$ — lattice structure: $P_\mathrm{I} = \Pi(L)$
- ξ-uncorrelated states: $\mathcal{D}_{\xi\text{-unc}} = \{\otimes_{X\in\xi}\varrho_X\}$ — $\upsilon \preceq \xi \iff \mathcal{D}_{\upsilon\text{-unc}} \subseteq \mathcal{D}_{\xi\text{-unc}}$
- ξ-separable states: $\mathcal{D}_{\xi\text{-sep}} = \operatorname{Conv}\mathcal{D}_{\xi\text{-unc}}$ — $\upsilon \preceq \xi \iff \mathcal{D}_{\upsilon\text{-sep}} \subseteq \mathcal{D}_{\xi\text{-sep}}$

Level II: multiple set partitions [3, 4, 5, 6]

- down-sets of partitions: $\boldsymbol{\xi} = \{\xi_1, \xi_2, \dots, \xi_{|\boldsymbol{\xi}|}\} \subseteq P_\mathrm{I}$ (closed downwards w.r.t. $\preceq$)
- partial order: $\boldsymbol{\upsilon} \preceq \boldsymbol{\xi}$ def.: $\boldsymbol{\upsilon} \subseteq \boldsymbol{\xi}$ — lattice structure: $P_\mathrm{II} = \mathcal{O}_\downarrow(P_\mathrm{I}) \setminus \{\emptyset\}$
- $\boldsymbol{\xi}$-uncorrelated states: $\mathcal{D}_{\boldsymbol{\xi}\text{-unc}} = \cup_{\xi\in\boldsymbol{\xi}}\mathcal{D}_{\xi\text{-unc}}$ — $\boldsymbol{\upsilon} \preceq \boldsymbol{\xi} \iff \mathcal{D}_{\boldsymbol{\upsilon}\text{-unc}} \subseteq \mathcal{D}_{\boldsymbol{\xi}\text{-unc}}$
- $\boldsymbol{\xi}$-separable states: $\mathcal{D}_{\boldsymbol{\xi}\text{-sep}} = \operatorname{Conv}\mathcal{D}_{\boldsymbol{\xi}\text{-unc}}$ — $\boldsymbol{\upsilon} \preceq \boldsymbol{\xi} \iff \mathcal{D}_{\boldsymbol{\upsilon}\text{-sep}} \subseteq \mathcal{D}_{\boldsymbol{\xi}\text{-sep}}$

Level III: classes [2, 4]

- up-sets of down-sets of partitions: $\underline{\boldsymbol{\xi}} = \{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots, \boldsymbol{\xi}_{|\underline{\boldsymbol{\xi}}|}\} \subseteq P_\mathrm{II}$ (closed upw. w.r.t. $\preceq$)
- partial order: $\underline{\boldsymbol{\upsilon}} \preceq \underline{\boldsymbol{\xi}}$ def.: $\underline{\boldsymbol{\upsilon}} \subseteq \underline{\boldsymbol{\xi}}$ — lattice structure: $P_\mathrm{III} = \mathcal{O}_\uparrow(P_\mathrm{II}) \setminus \{\emptyset\}$
- partial correlation classes [2]: all the possible intersections of $\mathcal{D}_{\boldsymbol{\xi}\text{-unc}}$

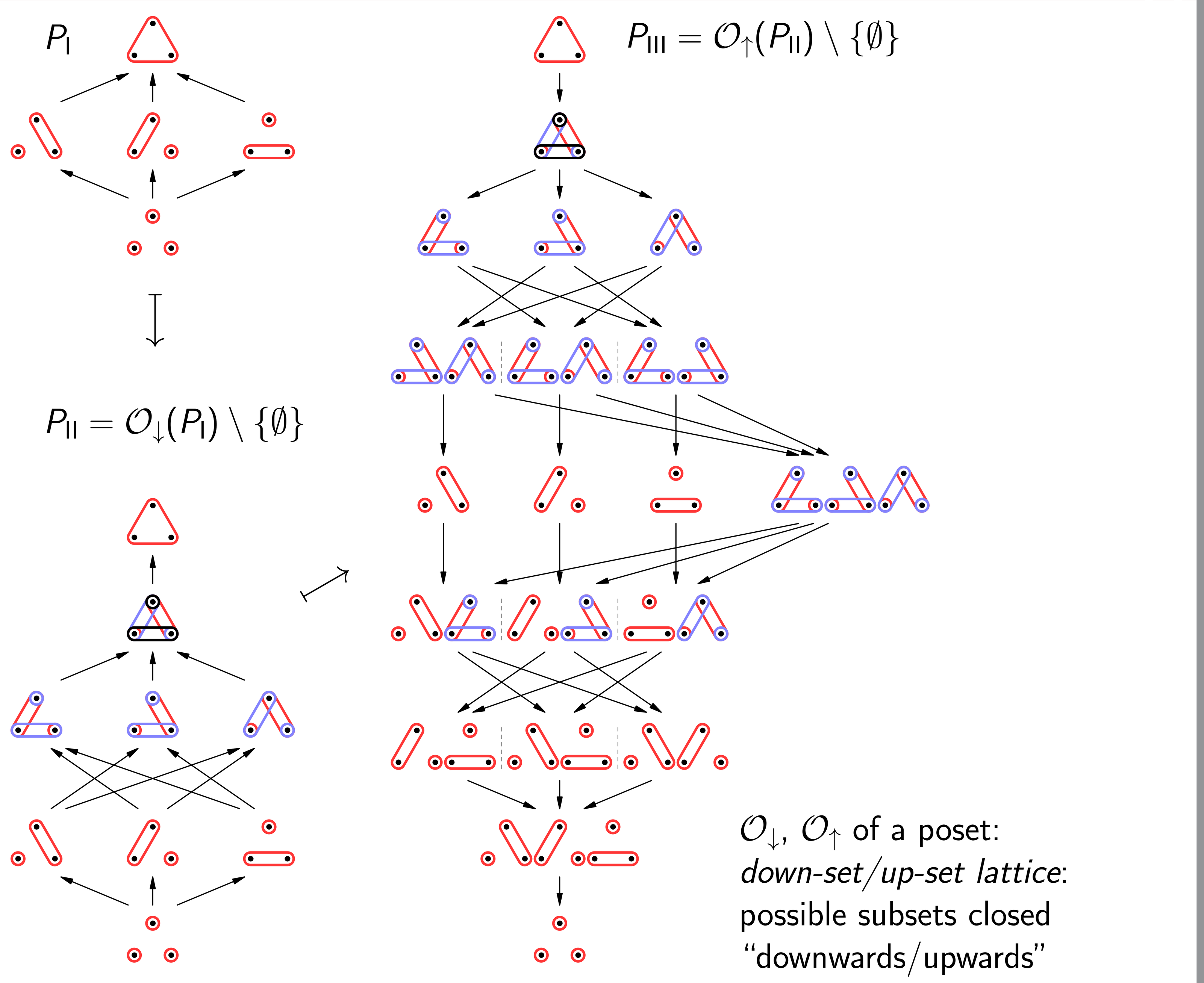
$$\mathcal{C}_{\underline{\boldsymbol{\xi}}\text{-unc}} = \bigcap_{\boldsymbol{\xi}\notin\underline{\boldsymbol{\xi}}} \overline{\mathcal{D}_{\boldsymbol{\xi}\text{-unc}}} \cap \bigcap_{\boldsymbol{\xi}\in\underline{\boldsymbol{\xi}}} \mathcal{D}_{\boldsymbol{\xi}\text{-unc}} \neq \emptyset \iff \text{principals: } \underline{\boldsymbol{\xi}} = \uparrow\{\downarrow\{\xi\}\}$$

- LO convertibility: if $\exists\varrho \in \mathcal{C}_{\underline{\boldsymbol{\upsilon}}\text{-unc}}$, $\exists\Lambda$ LO map such that $\Lambda(\varrho) \in \mathcal{C}_{\underline{\boldsymbol{\xi}}\text{-unc}}$ then $\underline{\boldsymbol{\upsilon}} \preceq \underline{\boldsymbol{\xi}}$
- partial separability classes [4]: all the possible intersections of $\mathcal{D}_{\boldsymbol{\xi}\text{-sep}}$

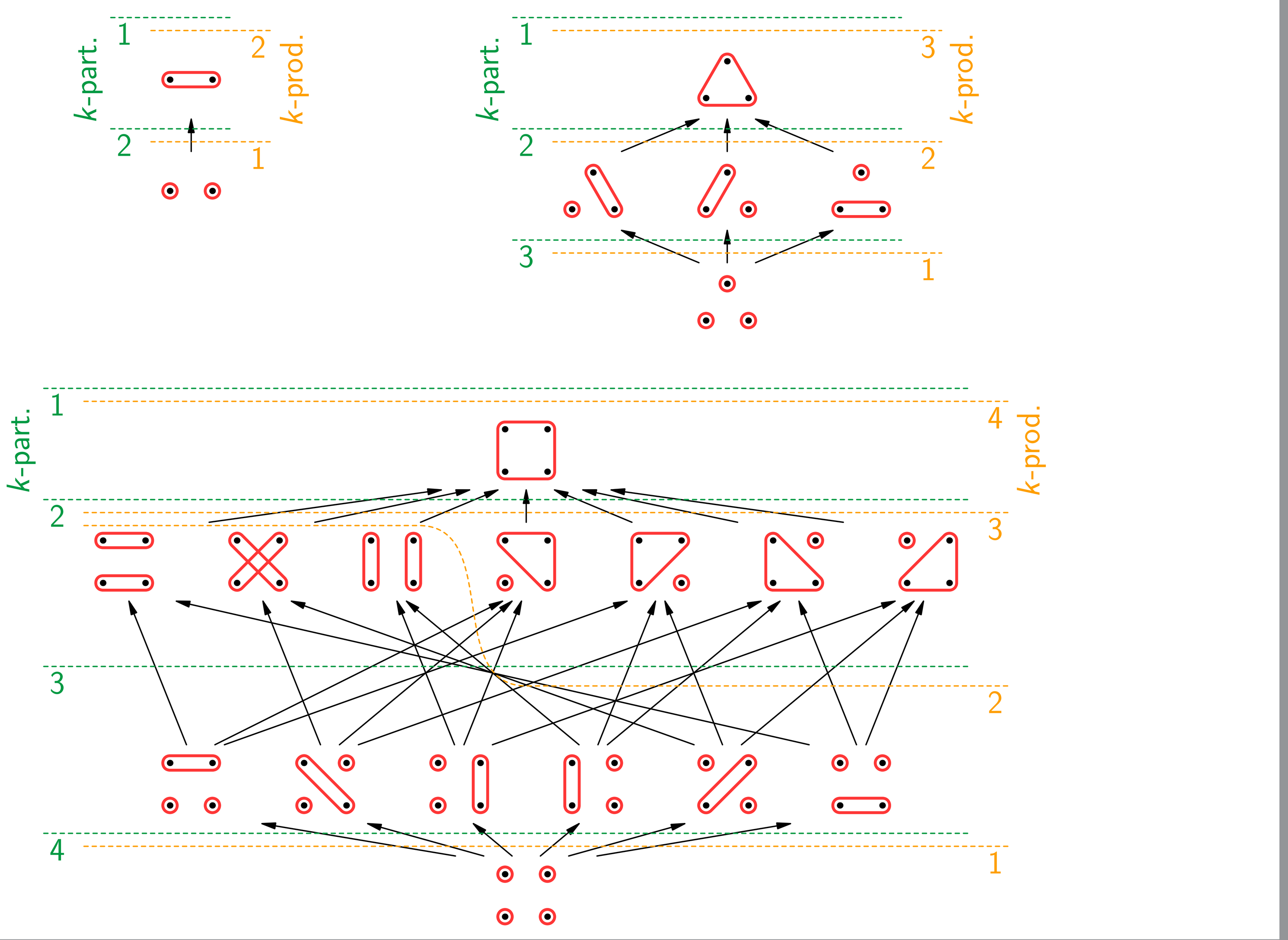
$$\mathcal{C}_{\underline{\boldsymbol{\xi}}\text{-sep}} = \bigcap_{\boldsymbol{\xi}\notin\underline{\boldsymbol{\xi}}} \overline{\mathcal{D}_{\boldsymbol{\xi}\text{-sep}}} \cap \bigcap_{\boldsymbol{\xi}\in\underline{\boldsymbol{\xi}}} \mathcal{D}_{\boldsymbol{\xi}\text{-sep}} \neq \emptyset \text{ for all } \underline{\boldsymbol{\xi}} \text{ (conjecture)}$$

- LOCC convertibility: if $\exists\varrho \in \mathcal{C}_{\underline{\boldsymbol{\upsilon}}\text{-sep}}$, $\exists\Lambda$ LOCC map such that $\Lambda(\varrho) \in \mathcal{C}_{\underline{\boldsymbol{\xi}}\text{-sep}}$ then $\underline{\boldsymbol{\upsilon}} \preceq \underline{\boldsymbol{\xi}}$

Illustrations: Structure for $n = 3$ [4]



Illustrations: k-partitionability and k'-producibility for $n = 2, 3, 4$ [3, 4]



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Quantification of properties [3, 4, 5]

- mixedness: von Neumann entropy $S(\varrho) = -\operatorname{tr}\varrho\ln\varrho$
- distinguishability: relative entropy $D(\varrho\|\sigma) = \operatorname{tr}\varrho(\ln\varrho - \ln\sigma)$

Level I: partitions [3, 4]

- ξ-correlation (ξ-mutual inf.): $C_\xi(\varrho) = \min_{\sigma\in\mathcal{D}_{\xi\text{-unc}}} D(\varrho\|\sigma) = \sum_{X\in\xi} S(\varrho_X) - S(\varrho)$
- ξ-entanglement: $E_\xi(\pi) = C_\xi|_\mathcal{P}(\pi)$, then convex roof
- corr./ent. monotones, faithful: $C_\xi(\varrho) = 0 \iff \varrho \in \mathcal{D}_{\xi\text{-unc}}$, $E_\xi(\varrho) = 0 \iff \varrho \in \mathcal{D}_{\xi\text{-sep}}$
- multipartite monotones: $\upsilon \preceq \xi \iff C_\upsilon \geq C_\xi$, $E_\upsilon \geq E_\xi$

Level II: multiple partitions [3, 4]

- $\boldsymbol{\xi}$-correlation: $C_{\boldsymbol{\xi}}(\varrho) = \min_{\sigma\in\mathcal{D}_{\boldsymbol{\xi}\text{-unc}}} D(\varrho\|\sigma) = \min_{\xi\in\boldsymbol{\xi}} C_\xi(\varrho)$
- $\boldsymbol{\xi}$-entanglement: $E_{\boldsymbol{\xi}}(\pi) = C_{\boldsymbol{\xi}}|_\mathcal{P}(\pi)$, then convex roof
- corr./ent. monotones, faithful: $C_{\boldsymbol{\xi}}(\varrho) = 0 \iff \varrho \in \mathcal{D}_{\boldsymbol{\xi}\text{-unc}}$, $E_{\boldsymbol{\xi}}(\varrho) = 0 \iff \varrho \in \mathcal{D}_{\boldsymbol{\xi}\text{-sep}}$
- multipartite monotones: $\boldsymbol{\upsilon} \preceq \boldsymbol{\xi} \iff C_{\boldsymbol{\upsilon}} \geq C_{\boldsymbol{\xi}}$, $E_{\boldsymbol{\upsilon}} \geq E_{\boldsymbol{\xi}}$

Structure of permutation invariant properties [1]

Level 0: subsystem sizes: $s(X) := |X|$, then elementwisely on P_0, then elementwisely on P_I, then...

$$\begin{array}{ccc}
(P_\mathrm{III}, \preceq) & \xrightarrow{s} & (\hat{P}_\mathrm{III}, \preceq) \\
\uparrow \mathcal{O}_\uparrow\setminus\{\emptyset\} & & \uparrow \mathcal{O}_\uparrow\setminus\{\emptyset\} \\
(P_\mathrm{II}, \preceq) & \xrightarrow{s} & (\hat{P}_\mathrm{II}, \preceq) \\
\uparrow \mathcal{O}_\downarrow\setminus\{\emptyset\} & & \uparrow \mathcal{O}_\downarrow\setminus\{\emptyset\} \\
(P_\mathrm{I}, \preceq) & \xrightarrow{s} & (\hat{P}_\mathrm{I}, \preceq) \\
\uparrow & & \uparrow \\
(P_0, \preceq) & \xrightarrow{s} & (\hat{P}_0, \preceq)
\end{array}$$

Level I: integer partitions *(Young diag.)*

- partition types of the system $\hat{\xi} = \{x_1, x_2, \dots, x_{|\hat{\xi}|}\} \in \hat{P}_\mathrm{I} := s(P_\mathrm{I})$ (multiset!)
- refinement: $\hat{\upsilon} \preceq \hat{\xi}$ def.: $\exists\upsilon \in s^{-1}(\hat{\upsilon}), \xi \in s^{-1}(\hat{\xi}) : \upsilon \preceq \xi$ — poset $\hat{P}_\mathrm{I}$
- Level II properties with $\boldsymbol{\xi} = \vee s^{-1}(\downarrow\{\hat{\xi}\})$ isom. $\hat{\upsilon} \preceq \hat{\xi} \iff \vee s^{-1}(\downarrow\{\hat{\upsilon}\}) \preceq \vee s^{-1}(\downarrow\{\hat{\xi}\})$
- state sets and measures as before

Level II: multiple integer partitions

- down-sets of int. part.s: $\hat{\boldsymbol{\xi}} = \{\hat{\xi}_1, \hat{\xi}_2, \dots, \hat{\xi}_{|\hat{\boldsymbol{\xi}}|}\} \subseteq \hat{P}_\mathrm{I}$
- Level II properties with $\boldsymbol{\xi} = \vee s^{-1}(\hat{\boldsymbol{\xi}})$ isomorphism $\hat{\boldsymbol{\upsilon}} \preceq \hat{\boldsymbol{\xi}} \iff \vee s^{-1}(\hat{\boldsymbol{\upsilon}}) \preceq \vee s^{-1}(\hat{\boldsymbol{\xi}})$
- state sets and measures as before

k-partitionability, k-producibility and k-stretchability [1]

- height, width and rank of Young diagrams

$$h(\hat{\xi}) = |\hat{\xi}| \qquad \hat{\boldsymbol{\xi}}_{k\text{-part}} = \{\hat{\xi} \in \hat{P}_\mathrm{I} \mid h(\hat{\xi}) \geq k\}$$
$$w(\hat{\xi}) = \max\hat{\xi} \qquad \hat{\boldsymbol{\xi}}_{k\text{-prod}} = \{\hat{\xi} \in \hat{P}_\mathrm{I} \mid w(\hat{\xi}) \leq k\}$$
$$r(\hat{\xi}) = w(\hat{\xi}) - h(\hat{\xi}) \qquad \hat{\boldsymbol{\xi}}_{k\text{-str}} = \{\hat{\xi} \in \hat{P}_\mathrm{I} \mid r(\hat{\xi}) \leq k\}$$

- down-sets: $\hat{\upsilon} \prec \hat{\xi} \Rightarrow h(\hat{\upsilon}) > h(\hat{\xi})$, $w(\hat{\upsilon}) \leq w(\hat{\xi})$, $r(\hat{\upsilon}) < r(\hat{\xi})$
- chains:

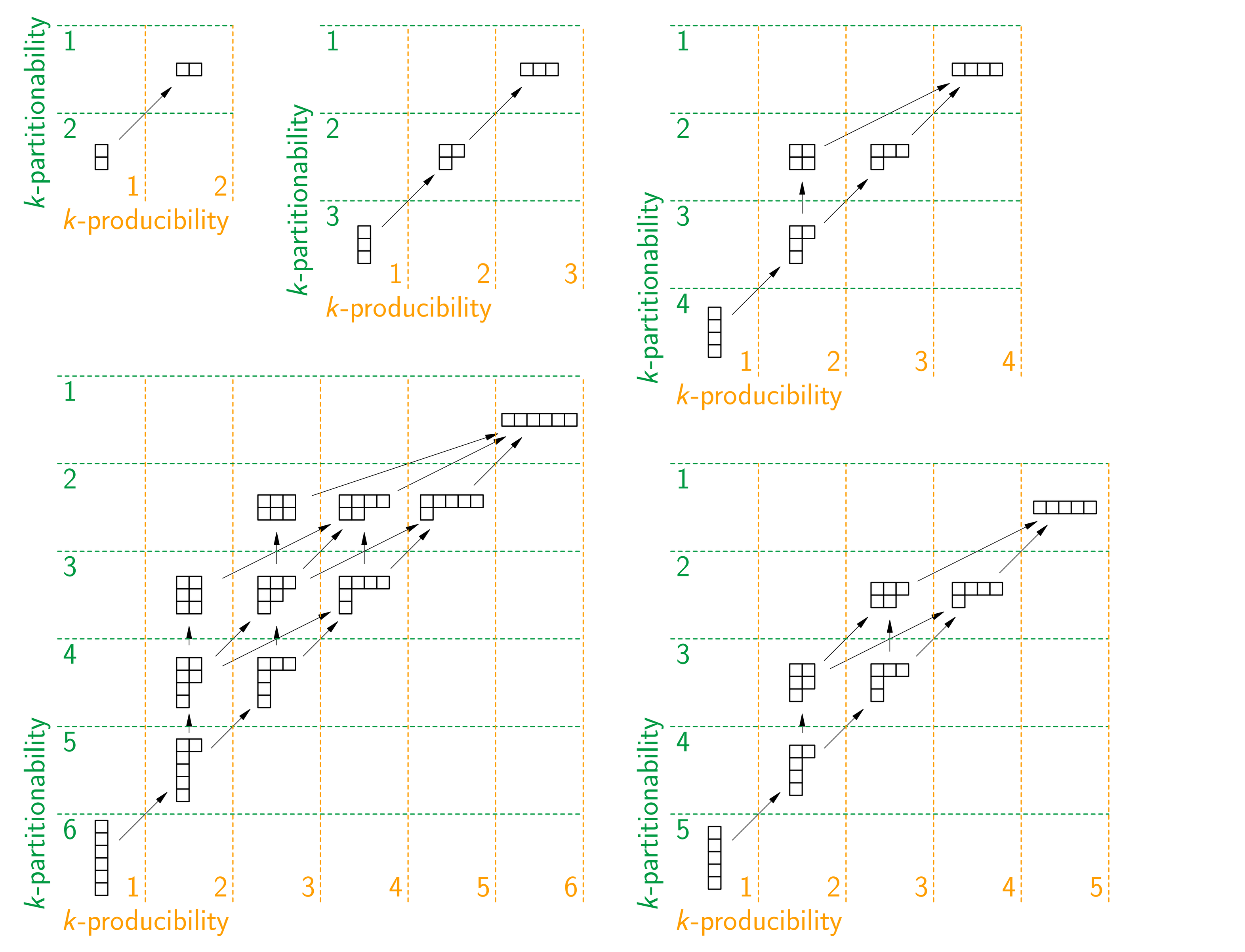
$$\hat{\boldsymbol{\xi}}_{l\text{-part}} \preceq \hat{\boldsymbol{\xi}}_{k\text{-part}} \iff l \geq k$$
$$\hat{\boldsymbol{\xi}}_{l\text{-part}} \preceq \hat{\boldsymbol{\xi}}_{k\text{-prod}} \iff l \leq k$$
$$\hat{\boldsymbol{\xi}}_{l\text{-str}} \preceq \hat{\boldsymbol{\xi}}_{k\text{-str}} \iff l \leq k$$

- bounds:

$$\hat{\boldsymbol{\xi}}_{k\text{-part}} \preceq \hat{\boldsymbol{\xi}}_{(n+1-k)\text{-prod}} \qquad \hat{\boldsymbol{\xi}}_{k\text{-part}} \preceq \hat{\boldsymbol{\xi}}_{(n+1-2k)\text{-str}}$$
$$\hat{\boldsymbol{\xi}}_{k\text{-prod}} \preceq \hat{\boldsymbol{\xi}}_{(\lceil n/k\rceil)\text{-part}} \qquad \hat{\boldsymbol{\xi}}_{k\text{-prod}} \preceq \hat{\boldsymbol{\xi}}_{(k-\lceil n/k\rceil)\text{-str}}$$
$$\hat{\boldsymbol{\xi}}_{k\text{-str}} \preceq \hat{\boldsymbol{\xi}}_{\frac{1}{2}(\lceil\sqrt{k^2+4n}\rceil-k)\text{-part}} \qquad \hat{\boldsymbol{\xi}}_{k\text{-str}} \preceq \hat{\boldsymbol{\xi}}_{\frac{1}{2}(n+1+k)\text{-prod}}$$

- duality by Young diag. conjugation: $h(\hat{\xi}^\dagger) = w(\hat{\xi})$, $w(\hat{\xi}^\dagger) = h(\hat{\xi})$, $r(\hat{\xi}^\dagger) = -r(\hat{\xi})$

Illustrations: Structure of permutation invariant properties for $n = 2 \dots 6$ [1]



Bibliography

[1] Sz. Szalay, *k-stretchability of entanglement, and the duality of k-separability and k-producibility*, Quantum **3**, 204 (2019).

[2] Sz. Szalay, *The classification of multipartite quantum correlation*, JPhysA **51**, 485302 (2018).

[3] Sz. Szalay, G. Barcza, T. Szilvási, L. Veis, Ö. Legeza, *Correlation theory of the chemical bond*, SciRep **7**, 2237 (2017).

[4] Sz. Szalay, *Multipartite entanglement measures*, PRA **92**, 042329 (2015).

[5] Sz. Szalay, Z. Kökényesi, *Partial separability revisited: Necessary and sufficient criteria*, PRA **86**, 032341 (2012).

[6] M. Seevinck, J. Uffink, *Partial separability and entanglement criteria for multiqubit quantum states*, PRA **78**, 032101 (2008).

[7] W. Dür, J. I. Cirac, *Classification of multiqubit mixed states: Separability and distillability properties*, PRA **61**, 042314 (2000).

Mail: szalay.szilard@wigner.mta.hu