Multipartite entanglement: the curious case of three qubits

Szilárd Szalay

Theoretical Solid State Physics Department, Wigner Research Centre for Physics, Budapest, Hungary.

September 24, 2015

Entanglement Day(s) 2015, Wigner FK SZFI, Budapest
info: `http://indico.kfki.hu/event/315/`

Motivation

Multipartite entanglement

- conceptionally richer than bipartite entanglement, paradigmatic new behaviour even for three qubits
- hard to grasp in general
- LOCC/SLOCC-like operative paradigms for entanglement become too complicated, coarse-grainings seem to be enforced
- genuine multipartite correlations can serve as powerful resources

Outline

Quantum system

States of a discrete quantum system

- state vector: $|\psi\rangle \in \mathcal{H}$ (normalized, $d = \dim \mathcal{H} < \infty$)
- pure state: $|\psi\rangle\langle\psi|$
  we are uncertain in the measurement outcomes,
  pure state encodes the *probabilities* of those
- mixed state (of an ensemble): $\varrho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$
  we are uncertain even in the (pure) state
- state space is a convex set, extremal points = pure states
- decomposition is not unique

Mixedness of quantum states

Measure of mixedness: entropies

- von Neumann entropy: $S(\varrho) = -\operatorname{tr} \varrho \ln \varrho$
- Rényi entropy: $S_{\mathrm{R},q}(\varrho) = \frac{1}{1-q} \ln \operatorname{tr} \varrho^q$
- Tsallis entropy: $S_{\mathrm{T},q}(\varrho) = \frac{1}{1-q} \big(\operatorname{tr} \varrho^q - 1\big)$
- Concurrence-squared: $C^2(\varrho) = 2 S_2^{\mathrm{Ts}}(\varrho) = 2\big(1 - \operatorname{tr} \varrho^2\big)$
- nonnegative, vanish iff ϱ pure
- Schur-concave: *entropy = mixedness*
- concave (Rényi $0 < q < 1$, Tsallis $0 < q$):
  increasing w.r.t. forgetting classical information
- Schumacher's noiseless coding thm:
  von Neumann entropy = quantum information content

Bengtsson, Życzkowski, *Geometry of Quantum States*, (Cambridge University Press 2006)

One qubit

Mixed state

- $\dim \mathcal{H} = 2$, state $\varrho = \frac{1}{2}(I + \mathbf{x}\boldsymbol{\sigma})$, in *Bloch sphere* $0 \leq \|\mathbf{x}\| \leq 1$
- eigenvalues: $\lambda_\pm = \frac{1}{2}\big(1 \pm \sqrt{1 - 4\det\varrho}\big) = \frac{1}{2}\big(1 \pm \|\mathbf{x}\|\big)$
- pure state: $\|\mathbf{x}\| = 1$, $|\psi\rangle = \mathrm{e}^{-i\varphi/2}\cos(\vartheta/2)|0\rangle + \mathrm{e}^{i\varphi/2}\sin(\vartheta/2)|0\rangle$
- mixedness: $C^2(\varrho) = 4\det\varrho = 1 - \|\mathbf{x}\|^2$, and $S(\varrho) = \mathcal{S}(C(\varrho))$ with $\mathcal{S}(x) = h\Big(\frac{1}{2}\big(1 + \sqrt{1 - x^2}\big)\Big)$, $h(x) = -x\log_2 x - (1-x)\log_2(1-x)$

SL structure

- two-form $\varepsilon \in \mathrm{Lin}(\mathcal{H} \to \mathcal{H}^*)$ with antisymmetric $\varepsilon_{ii'} = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$
- action: $\langle\tilde{\psi}| = \varepsilon|\psi\rangle$, $|\psi\rangle \mapsto |\tilde{\psi}\rangle = \langle\tilde{\psi}|^*$, $\varrho \mapsto \tilde{\varrho} = (\varepsilon\varrho\varepsilon^\dagger)^*$
  on Bloch vector: $\mathbf{x} \mapsto \tilde{\mathbf{x}} = -\mathbf{x}$ (splin flip)
- transformation: $A^{\mathrm{t}}\varepsilon A = (\det A)\varepsilon$, so $\mathrm{SL}(2,\mathbb{C})$-invariance
- mixedness: $C^2(\varrho) = 4\det\varrho = 2\,\mathrm{tr}\,\varrho\tilde{\varrho}$

Bipartite systems and entanglement

Pure States

- $|\psi\rangle \in \mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$
- There are uncorrelated, separable states: $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$.
- Nonclassical: there are also correlated ones, entangled.
  Then measurement on subsystem 1 "causes"$^?$ the collapse of the state of subsystem 2. (worry of EPR)
- states of subsystems (e.g., $\mathrm{tr}_2 |\psi\rangle\langle\psi|$) are not necessarily pure
- $|\psi\rangle$ is entangled if (and only if) $\mathrm{tr}_2 |\psi\rangle\langle\psi|$ and $\mathrm{tr}_1 |\psi\rangle\langle\psi|$ are mixed
  In this case, *"the best possible knowledge of the whole does not involve the best possible knowledge of its parts."* (Schrödinger)

Bipartite systems and entanglement

Pure States

- There are uncorrelated, separable states: $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$.
- Nonclassical: there are also correlated ones, entangled.
- decision of separability is simple: $|\psi\rangle$ is separable iff $\mathrm{tr}_2 |\psi\rangle\langle\psi|$ is pure

Mixed States

- a mixed state is separable if there exists separable decomposition:

$$\varrho = \sum_i p_i \big(|\psi_{1,i}\rangle \otimes |\psi_{2,i}\rangle\big)\big(\langle\psi_{1,i}| \otimes \langle\psi_{2,i}|\big),$$

- classically correlated sources produce states of this kind (Werner) can be prepared by Local Operations and Classical Communication
- the others are entangled
- the decomposition is not unique
- decision of separability is difficult

Bipartite systems and entanglement – Pure states

Pure States

- separable: $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$, else it is entangled
- decision of separability is simple: $|\psi\rangle$ is separable iff $\mathrm{tr}_2 |\psi\rangle\langle\psi|$ is pure

Measure of entanglement

- von Neumann entropy of entanglement: $E(|\psi\rangle) = S(\mathrm{tr}_2 |\psi\rangle\langle\psi|)$
- Rényi entropy of entanglement: $E_{\mathrm{R},q<1}(|\psi\rangle) = S_{\mathrm{R},q}(\mathrm{tr}_2 |\psi\rangle\langle\psi|)$
- Tsallis entropy of entanglement: $E_{\mathrm{T},q}(|\psi\rangle) = S_{\mathrm{T},q}(\mathrm{tr}_2 |\psi\rangle\langle\psi|)$
- Concurrence of entanglement: $E_{\mathrm{C}}(|\psi\rangle) = C(\mathrm{tr}_2 |\psi\rangle\langle\psi|)$
- the *mixedness of the subsystem* is a good *measure of entanglement* (entanglement monotone: nonincreasing on average w.r.t. pure LOCC
  Vidal: isometry-invariant concave function of $\mathrm{tr}_2 |\psi\rangle\langle\psi|$)
- vanish exactly for separable states

Bipartite systems and entanglement – Mixed states

Mixed States

- separable: $\varrho = \sum_i p_i \big(|\psi_{1,i}\rangle \otimes |\psi_{2,i}\rangle\big)\big(\langle\psi_{1,i}| \otimes \langle\psi_{2,i}|\big)$, else entangled
- decision of separability is difficult

Measure of entanglement

- the *average entanglement of the optimal decomposition* (convex roof extension of ent. entropy, entanglement of formation)

$$E(|\psi\rangle) = S(\mathrm{tr}_2 |\psi\rangle\langle\psi|) \quad \rightsquigarrow \quad E^{\cup}(\varrho) = \min_{\varrho=\sum_i p_i|\psi_i\rangle\langle\psi_i|} \sum_i p_i E(|\psi_i\rangle)$$

  is a good *measure of entanglement* (entanglement monotone)
- simillarly, Rényi, Tsallis entanglement of formation $E^{\cup}_{\mathrm{R},q<1}$, $E^{\cup}_{\mathrm{T},q}$, Concurrence of formation $E^{\cup}_{\mathrm{C}}$
- vanish exactly for separable states

Two qubits

Pure States

- $\dim\mathcal{H}_1 = \dim\mathcal{H}_2 = 2$, state: $|\psi\rangle = \sum_{i,j=0}^{1}\psi^{ij}|ij\rangle \in \mathcal{H}_{12} = \mathcal{H}_1\otimes\mathcal{H}_2$
- $E_{\mathrm{C}}(|\psi\rangle) = C(\mathrm{tr}_2\,|\psi\rangle\langle\psi|)$ and $E(|\psi\rangle) = \mathcal{S}\big(E_{\mathrm{C}}(|\psi\rangle)\big)$
- we had: $E_{\mathrm{C}}^2(\psi) = 4\det(\mathrm{tr}_2\,|\psi\rangle\langle\psi|)$
  easy: $E_{\mathrm{C}}(\psi) = 2|\det\psi| = 2|\psi^{00}\psi^{11} - \psi^{01}\psi^{10}|$
  $$= |\langle\tilde{\psi}|\psi\rangle| = \varepsilon_{ii'}\varepsilon_{jj'}\psi^{ij}\psi^{i'j'} = \psi^{\cdot\cdot}\psi^{\cdot\cdot},$$
  (with the two-qubit spin-flip $\langle\tilde{\psi}| = \varepsilon\otimes\varepsilon|\psi\rangle$)
- hard: $E_C^{\cup}(\varrho) = \big(\lambda_1^{\downarrow} - \lambda_2^{\downarrow} - \lambda_3^{\downarrow} - \lambda_4^{\downarrow}\big)^{+}$, and $E^{\cup}(\varrho) = \mathcal{S}\big(E_C^{\cup}(\varrho)\big)$,
  where $\lambda_i^{\downarrow}$s are the decreasingly ordered eigenvalues of the positive matrix $\sqrt{\sqrt{\varrho}\tilde{\varrho}\sqrt{\varrho}}$, written with the spin-flip $\tilde{\varrho} = (\varepsilon\otimes\varepsilon\varrho\varepsilon^{\dagger}\otimes\varepsilon^{\dagger})^{*}$.
  This is called Wootters' concurrence. ($\mathrm{SL}(2,\mathbb{C})^{\times 2}$-invariant)

Wootters, PRL **80**, 2245 (1998)

Two-qubit entanglement inside the three-qubit system

Entanglement monogamy

- $\dim \mathcal{H}_1 = \dim \mathcal{H}_2 = \dim \mathcal{H}_3 = 2$,
  state: $|\psi\rangle = \sum_{i,j,k=0}^{1} \psi^{ijk} |ijk\rangle \in \mathcal{H}_{123} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$
- states of subsystems: $\varrho_{bc} = \mathrm{tr}_a |\psi\rangle\langle\psi|$ and $\varrho_a = \mathrm{tr}_{bc} |\psi\rangle\langle\psi|$
- restriction on bipartite entanglement (monogamy, CKW ineqality):

$$E_C^{\cup}(\varrho_{ab})^2 + E_C^{\cup}(\varrho_{ac})^2 \leq C^2(\varrho_a)$$

- GHZ: $|\psi_{\mathrm{GHZ}}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$: $\quad 0 + 0 < 1$
  W: $|\psi_{\mathrm{W}}\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$: $\quad 4/9 + 4/9 = 8/9$
- there's no such restriction for correlation in classical systems
- moreover, holds for n qubits $\sum_{b \neq a} E_C^{\cup}(\varrho_{ab})^2 \leq C^2(\varrho_a)$
- also, entanglement restricts classical correlation, and vice versa

Coffman, Kundu, Wootters, PRL **61** 052306 (2000)

Osborne, Verstraete, PRL **96** 220503 (2006)

Koashi, Winter, PRA **69** 022309 (2004)

Two-qubit entanglement inside the three-qubit system

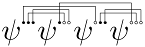
Distributed entanglement

- append the monogamy inequality with the difference:

$$E_C^{\cup}(\varrho_{ab})^2 + E_C^{\cup}(\varrho_{ac})^2 + \tau(\psi) = C^2(\varrho_a)$$

- three-tangle: $\tau(\psi) = 4|\mathrm{Det}\,\psi|$, with Cayley's hyperdeterminant

$$\mathrm{Det}\,\psi = -\frac{1}{2}\varepsilon_{ii'}\varepsilon_{jj'}\varepsilon_{kk'}\varepsilon_{ll'}\varepsilon_{mm'}\varepsilon_{nn'}\psi^{ikl}\psi^{jk'l'}\psi^{i'mn}\psi^{j'm'n'} = -\frac{1}{2}$$



$$= \psi^{000}\psi^{111}\psi^{000}\psi^{111} + \psi^{110}\psi^{001}\psi^{110}\psi^{001} + \psi^{101}\psi^{010}\psi^{101}\psi^{010} + \psi^{011}\psi^{100}\psi^{011}\psi^{100}$$
$$- 2\big(\psi^{000}\psi^{111}\psi^{110}\psi^{001} + \psi^{000}\psi^{111}\psi^{101}\psi^{010} + \psi^{000}\psi^{111}\psi^{011}\psi^{100}$$
$$+ \psi^{110}\psi^{001}\psi^{101}\psi^{010} + \psi^{110}\psi^{001}\psi^{011}\psi^{100}$$
$$+ \psi^{101}\psi^{010}\psi^{011}\psi^{100}\big)$$
$$+ 4\big(\psi^{000}\psi^{110}\psi^{101}\psi^{011} + \psi^{111}\psi^{001}\psi^{010}\psi^{100}\big),$$

- $\mathrm{SL}(2,\mathbb{C})^{\times 3}$- and permutation-invariant, entanglement monotone

Coffman, Kundu, Wootters, PRL **61** 052306 (2000)

Some ways of classificaiton

- Local Operation and Classical Communication (LOCC), $\varrho \cong_{\text{LOCC}} \omega$: they can be converted into each other by means of LOCC
- LOCC classification for pure states: $|\psi\rangle \cong_{\text{LOCC}} |\varphi\rangle$ iff $|\psi\rangle \cong_{\text{LU}} |\varphi\rangle$: Local Unitary equivalent, $|\varphi\rangle = U_1 \otimes \cdots \otimes U_n |\psi\rangle$, $U_i \in \mathrm{U}(\mathcal{H}_i)$
- Stochastic LOCC (SLOCC), $\varrho \cong_{\text{SLOCC}} \omega$: they can be converted into each other by means of LOCC with nonzero probability of success
- SLOCC classif. for pure states: $|\psi\rangle \cong_{\text{SLOCC}} |\varphi\rangle$ iff $|\psi\rangle \cong_{\text{LGL}} |\varphi\rangle$: Local General Linear equiv., $|\varphi\rangle = \frac{1}{p_{\text{succ}}} G_1 \otimes \cdots \otimes G_n |\psi\rangle$, $G_i \in \mathrm{GL}(\mathcal{H}_i)$
- Partial Separability (PS), $\varrho \cong_{\text{PS}} \omega$: see in my next talk
- PS classification for pure states: $|\psi\rangle \cong_{\text{PS}} |\varphi\rangle$ iff the finest way of decomposition to tensor product form are the same

LOCC classification of two- and three-qubit pure states

can be given by LU canonical form:

- two qubits: Schmidt decomposition $|\psi\rangle = \sqrt{\eta_0}|00\rangle + \sqrt{\eta_1}|11\rangle$, with $\eta_0 \geq \eta_1 \geq 0$, (normalized: 1-parameter set of LOCC classes)
- LOCC conversion (Nielsen): $|\psi\rangle \mapsto |\psi'\rangle$ iff $\eta_0 \leq \eta_0'$
- three qubits: canonical decomp. $|\psi\rangle = \sum_i \sqrt{\eta_i}|\varphi_{1,i}\rangle \otimes |\varphi_{2,i}\rangle \otimes |\varphi_{3,i}\rangle$, not locally orthogonal vectors, border rank problem...
- three qubits: generalized Schmidt decomposition $|\psi\rangle = \sqrt{\eta_0}|000\rangle + \sqrt{\eta_1}\mathrm{e}^{i\phi}|100\rangle + \sqrt{\eta_2}|101\rangle + \sqrt{\eta_3}|110\rangle + \sqrt{\eta_4}|111\rangle$, $0 \leq \eta_i$, $0 \leq \phi \leq \pi$, 5-parameter set of LOCC classes

can also be given by a sufficient set of LU invariants
even max. entangled state sets (w.r.t. LOCC) show structure too involved

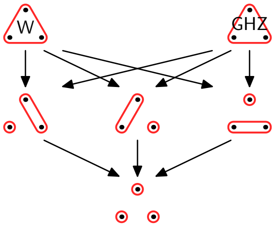
Acín, Andrianov, Costa, Jané, Latorre, Tarrach, PRL **85** 1560 (2000)

SLOCC classification of two- and three-qubit pure states

- two qubits: 2 classes, representative elements
  entangled: $|\psi_{\text{ent}}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
  separable: $|\psi_{\text{sep}}\rangle = |00\rangle$
- local selective measurements: entangled $\rightarrow$ separable
- three qubits: 6 classes, representative elements
  GHZ: $|\psi_{\text{GHZ}}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$
  W: $|\psi_{\text{W}}\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$
  1|23-biseparable: $|\psi_{1|23}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |011\rangle)\dots$
  fully separable: $|\psi_{1|2|3}\rangle = |000\rangle$
- principles: local rank and minimal number of product terms in decomposition are invariant w.r.t. SLOCC
- GHZ class: $\tau(\psi) > 0$
- generalizations: two qubits with a qutrit or qu4it: discrete classes
  four qubits: nine families of continuously parametrized classes

SLOCC classification of two- and three-qubit pure states

- two qubits: 2 classes, representative elements
  entangled: $|\psi_{\text{ent}}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
  separable: $|\psi_{\text{sep}}\rangle = |00\rangle$
- local selective measurements: entangled $\rightarrow$ separable
- three qubits: 6 classes, local selective measurements:



$\longleftrightarrow$ local rank $= (2, 2, 2)$

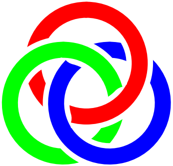
$\longleftrightarrow$ local rank $= (1, 2, 2) \dots$

$\longleftrightarrow$ local rank $= (1, 1, 1)$

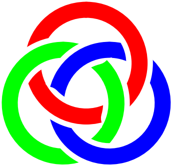
- GHZ class: $\tau(\psi) > 0$
- generalizations: two qubits with a qutrit or qu4it: discrete classes
  four qubits: nine families of continuously parametrized classes

SLOCC classification of two- and three-qubit pure states

- two qubits: 2 classes, representative elements
  entangled: $|\psi_{\text{ent}}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
  separable: $|\psi_{\text{sep}}\rangle = |00\rangle$
- local selective measurements: entangled $\rightarrow$ separable
- separable two-qubit subsystems (only) in the GHZ class



$|\psi_{\text{W}}\rangle = \frac{1}{\sqrt{3}}\big(|100\rangle + |010\rangle + |001\rangle\big)$



$|\psi_{\text{GHZ}}\rangle = \frac{1}{\sqrt{2}}\big(|000\rangle + |111\rangle\big)$

- generalizations: two qubits with a qutrit or qu4it: discrete classes
  four qubits: nine families of continuously parametrized classes

+1: FTS classification of three-qubit pure states

- $\mathcal{H}_{123} \cong \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathfrak{M}(\mathcal{J}) \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathcal{J} \oplus \mathcal{J}$ Freudenthal Triple System over the cubic Jordan algebra $\mathcal{J} \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$
- each element has an LGL-invariant "Freudenthal rank", given by the vanishing of the $\mathrm{SL}(2,\mathbb{C})$-covariants

$$q(\psi) = \psi\psi\psi\psi$$

$$T(\psi) = \psi\psi\psi = \psi\psi\psi = \psi\psi\psi$$

$$\gamma_1(\psi) = \psi\psi, \qquad \gamma_2(\psi) = \psi\psi, \qquad \gamma_3(\psi) = \psi\psi$$

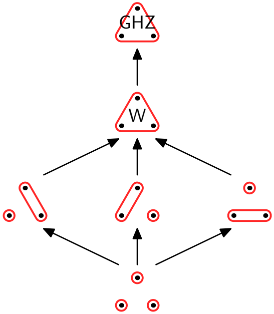
$$\Upsilon_\phi(\psi) = \psi\psi\phi + \psi\psi\phi + \psi\psi\phi$$

- states are SLOCC equivalents iff Freudenthal ranks are the same, 4: GHZ, 3: W, 2: biseparable, 1: fully separable
- classification: same as SLOCC, but different hierarchy
- generalization: only for some special number of special dimensions

Borsten, Dahanayake, Duff, Rubens, Ebrahim, PRA **80** 032326 (2009)

$+1$: FTS classification of three-qubit pure states

- $\mathcal{H}_{123} \cong \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathfrak{M}(\mathcal{J}) \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathcal{J} \oplus \mathcal{J}$ Freudenthal Triple System over the cubic Jordan algebra $\mathcal{J} \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$



$\longleftrightarrow$ FTS rank $= 4$

$\longleftrightarrow$ FTS rank $= 3$

$\longleftrightarrow$ FTS rank $= 2$

$\longleftrightarrow$ FTS rank $= 1$

- classification: same as SLOCC, but different hierarchy
- generalization: only for some special number of special dimensions

Borsten, Dahanayake, Duff, Rubens, Ebrahim, PRA **80** 032326 (2009)

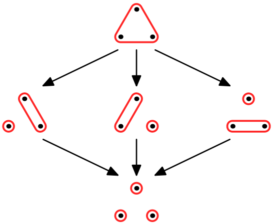
PS classification of two- and three-qubit pure states

- two qubits: 2 classes, representative elements
  entangled: $|\psi_{12}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
  separable: $|\psi_{1|2}\rangle = |00\rangle$
- local selective measurements: entangled $\rightarrow$ separable
- three qubits: 5 classes, representative elements
  fully entangled: $|\psi_{123}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$
  1|23-biseparable: $|\psi_{1|23}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |011\rangle)\dots$
  fully separable: $|\psi_{1|2|3}\rangle = |000\rangle$

- can be given for arbitrary number of arbitrary subsystems, discrete

PS classification of two- and three-qubit pure states

- two qubits: 2 classes, representative elements
  entangled: $|\psi_{12}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
  separable: $|\psi_{1|2}\rangle = |00\rangle$
- local selective measurements: entangled $\rightarrow$ separable
- three qubits: 5 classes, local selective measurements:



- can be given for arbitrary number of arbitrary subsystems, discrete

Classifications of two- and three-qubit mixed states

I'm pretty sure that we won't have time for this one.

Szalay, arXiv:1503.06071 [quant-ph] (2015)

GHZ experiment

- three spin-$\frac{1}{2}$ particles, three detectors with two settings ($\{0,1\}$) each, with two outcomes ($\{0,1\}$) each
- settings: restricted to odd number of 0s
- outcomes: $\{011,101,110\} \mapsto \{011,101,110,000\}$ (odd number of 0)
  $\{000\} \mapsto \{100,010,001,111\}$ (even number of 0)
- EPR-like argument: outcomes should be stored in LHVs, but LHVs cannot be given. (Local Hidden Variable)
- quantum mechanical description: state: $|\psi'_{\mathrm{GHZ}}\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$, simultaneous eigenvector of observables: $\sigma_x \otimes \sigma_y \otimes \sigma_y$, $\sigma_y \otimes \sigma_x \otimes \sigma_y$, $\sigma_y \otimes \sigma_y \otimes \sigma_x$, $\sigma_x \otimes \sigma_x \otimes \sigma_x$, with eigenvalues $1,1,1,-1$, respectively.
- not only statistical, but absolute contradiction with LHVM!

Mermin, AmJPhys **58**, 731 (1990)

Classical nonlocal games

Cooperative games of incomplete information

- players (Alice, Béla,. . .) with prearranged collective strategy,
- referee, poses questions $q \in Q$, $r \in R$,. . . (with known prob. p)
- players answer $a \in A$, $b \in B$,. . . (without communication and without knowing the others' questions)
- referee evaluates (predicate V on $A \times B \times \dots \times Q \times R \times \dots$)
- deterministic strategy: functions $a : Q \to A$, $b : R \to B$,. . .
- classical value of nonlocal game (best winning probability):

$$\omega_c = \max_{a(q),b(r),\dots} \sum_{q,r,\dots} p(q,r,\dots) V(a(q), b(r), \dots | q, r, \dots)$$

- (probabilistic strategies: convex combinations of deterministic ones)

Cleve, Høyer, Toner, Watrous, arXiv:0404076v2 [quant-ph] (bugfixed IEEE Conf paper (2004))

Quantum nonlocal games

Cooperative games of incomplete information + shared entanglement

- same situation as before, but entangled multipratite sys. shared, $|\psi\rangle$
- players have POVMs for each question (strategy),
  Alice: $\forall q \in Q : X_q = \{X_q^a | \sum_{a\in A} X_q^a = I\}$,
  Béla: $\forall r \in R : Y_r = \{Y_r^b | \sum_{b\in B} Y_r^b = I\}, \dots$
- they perform the measurement corresponding to the questions and answer the outcomes
- quantum value of nonlocal game ($\sim$best winning probability):

$$\omega_{\mathbf{q}} = \sup_{\{X_q\},\{Y_r\},\dots} \sum_{q,r,\dots} p(q,r,\dots) \sum_{a,b,\dots} \langle\psi| X_q^a \otimes Y_r^b \otimes \dots |\psi\rangle V(a,b,\dots|q,r,\dots)$$

Cleve, Høyer, Toner, Watrous, arXiv:0404076v2 [quant-ph] (bugfixed IEEE Conf paper (2004))

Quantum nonlocal game of two players

CHSH game

- players: Alice, Bob; questions: $\{0,1\}$; answers: $\{0,1\}$
- game: p uniform, $V(a,b|q,r)=1$ if $a\oplus b=q\wedge r$, otherwise 0
  that is, $11\mapsto$ odd 1s, $00,01,10\mapsto$ even 1s,
- classical value: $\omega_{\mathrm{c}}=3/4=0.75$
  contradictory system:
  $a(0)\oplus b(0)=0$
  $a(0)\oplus b(1)=0$
  $a(1)\oplus b(0)=0$
  $a(1)\oplus b(1)=1$
  best they can do: say always 0

Cleve, Høyer, Toner, Watrous, arXiv:0404076v2 [quant-ph] (bugfixed IEEE Conf paper (2004))

Quantum nonlocal game of two players

CHSH game

- players: Alice, Bob; questions: $\{0,1\}$; answers: $\{0,1\}$
- game: p uniform, $V(a,b|q,r) = 1$ if $a \oplus b = q \wedge r$, otherwise 0
  that is, $11 \mapsto$ odd 1s, $00,01,10 \mapsto$ even 1s,
- classical value: $\omega_\text{c} = 3/4 = 0.75$
- quantum value: $\omega_\text{q} = \cos^2(\pi/8) = \frac{1}{2} + \frac{1}{2\sqrt{2}} \approx 0.85$
  share $|\psi_\text{B}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ states,
  X_0: proj. spin meas. along $(0,0,1)$ direction (up: 0; down: 1)
  X_1: proj. spin meas. along $(1,0,0)$ direction
  Y_0: proj. spin meas. along $(1,0,1)/\sqrt{2}$ direction
  Y_1: proj. spin meas. along $(1,0,-1)/\sqrt{2}$ direction
  (Tsirelson bound: optimal for shared qubits)

Cleve, Høyer, Toner, Watrous, arXiv:0404076v2 [quant-ph] (bugfixed IEEE Conf paper (2004))

Quantum nonlocal game of three players

Three-qubit game

- players: Alice, Bob, Charlie; questions: $\{0, 1\}$; answers: $\{0, 1\}$
- questions: restricted to odd number of 0s
- game: p uniform,
  $V(a, b, c|q, r, s) = 1$ if $a \oplus b \oplus c = q \vee r \vee s$, otherwise 0,
  that is, $000 \mapsto$ even 1s, $011, \ldots \mapsto$ odd 1s,
- classical value: $\omega_{\mathrm{c}} = 3/4$
  contradictory system:
  $a(0) \oplus b(0) \oplus c(0) = 0$
  $a(0) \oplus b(1) \oplus c(1) = 1$
  $a(1) \oplus b(0) \oplus c(1) = 1$
  $a(1) \oplus b(1) \oplus c(0) = 1$
  best they can do: say always 1

Quantum nonlocal game of three players

Three-qubit game

- players: Alice, Bob, Charlie; questions: $\{0,1\}$; answers: $\{0,1\}$
- questions: restricted to odd number of 0s
- game: p uniform,
  $V(a,b,c|q,r,s) = 1$ if $a \oplus b \oplus c = q \vee r \vee s$, otherwise 0,
  that is, $000 \mapsto$ even 1s, $011, \dots \mapsto$ odd 1s,
- classical value: $\omega_\mathrm{c} = 3/4$
- quantum value: $\omega_\mathrm{q} = 1$!
  share $|\psi''_\mathrm{GHZ}\rangle = \frac{1}{2}(|000\rangle - |011\rangle - |101\rangle - |110\rangle) = (PH)^{\otimes 3}|\psi_\mathrm{GHZ}\rangle$
  X_0, Y_0: proj. spin-z meas. ($|0\rangle, |1\rangle$ basis)
  X_1, Y_1: proj. spin-x meas. $((|0\rangle + |1\rangle)/\sqrt{2}, (|0\rangle - |1\rangle)/\sqrt{2})$
  $qrs = 000$: even 1s: always win
  $qrs = 011$: odd 1s: always win,
  check: $I \otimes H \otimes H|\psi''_\mathrm{GHZ}\rangle = \frac{1}{2}(|001\rangle + |010\rangle - |100\rangle + |110\rangle)$

Quantum nonlocal game of three players

Three-qubit game with other kinds of entanglement

- game: $000 \mapsto$ even 1s, $011, \ldots \mapsto$ odd 1s,
- classical value: $\omega_{\mathrm{c}} = 3/4$
- quantum value with shared GHZ-class states (FTS rank-4): $\omega_{\mathrm{GHZ}} = 1$
- from rank conditions: for shared W-class states (FTS rank-3), $\omega_{\mathrm{W}} < 1$
- winning GHZ strategy for W states: $7/8 = 0.875 \leq \omega_{\mathrm{W}}$
- for shared biesparable states (FTS rank-2):
  Tsirelson-type construction, $\omega_{\mathrm{bisep}} \leq \frac{1}{2} + \frac{1}{2\sqrt{2}} \approx 0.85$
  equivalent to CHSH game, so this bound is attained
- uncorrelated states (FTS rank-1) is like having nothing, $\omega_{\mathrm{sep}} = \omega_{\mathrm{c}}$

$$\omega_{\mathrm{c}} = \omega_{\mathrm{sep}} = 3/4 < \omega_{\mathrm{bisep}} = \frac{1}{2} + \frac{1}{2\sqrt{2}} < 7/8 \leq \omega_{\mathrm{W}} < 1 = \omega_{\mathrm{GHZ}}$$

Borsten, JPhysA **46** 455303 (2013)

Summary

Multipartite entanglement

- conceptionally richer than bipartite entanglement
  paradigmatic new behaviour even for three qubits
  - monogamy: restriction on entanglements inside subsystems
  - classificaiton issues: LOCC/SLOCC, FTS, PS
  - GHZ correlations: absolute contradiction with LHVM
  - quantum nonlocal games: resources of different values
    (not LOCC/SLOCC situation, but still local, operative approach)
- hard to grasp in general
- LOCC/SLOCC-like operative paradigm for entanglement
  become too complicated, coarse-grainings seem to be enforced

Thank you for your attention!

This project was supported by
the Hungarian Scientific Research Fund (project ID: OTKA-K100908) and
the "Lendület" program of the Hungarian Academy of Sciences (project ID: 81010-00).