Let us have the finite dimensional Hilbert space $\mathcal{H}$ and the density operators $\mathcal{D} \subset \operatorname{Lin}\mathcal{H}$ over it. Let us have two POVMs (*positive operator valued measures*)

$$E = \Big\{E_i \geq 0 \,\Big|\, i \in I,\ \sum_{i\in I} E_i = \mathrm{I}\Big\} \subset \operatorname{Lin}\mathcal{H} \text{ and}$$
$$F = \Big\{F_j \geq 0 \,\Big|\, j \in J,\ \sum_{j\in J} F_j = \mathrm{I}\Big\} \subset \operatorname{Lin}\mathcal{H}.$$

We have the following relations between them:
(COM) E and F *commute*, if $\forall i \in I$ and $\forall j \in J$ we have $[E_i, F_j] = 0$.
(ND) E *does not disturb* F, if there exists an instrument $\{\Phi_{E,i}\}$, which implements E (that is, $\forall i \in I$ and $\forall \varrho \in \mathcal{D}$ we have $\operatorname{tr}\Phi_{E,i}(\varrho) = \operatorname{tr} E_i\varrho$) and which does not disturb F (that is, $\forall j \in J$ and $\forall \varrho \in \mathcal{D}$ we have $\operatorname{tr}\sum_i \Phi_{E,i}(\varrho)F_j = \operatorname{tr}\varrho F_j$).
(JM) E and F are *jointly measurable*, if there exists a POVM $G = \{G_{i,j} \geq 0 \mid i \in I, j \in J, \sum_{i\in I, j\in J} G_{i,j} = \mathrm{I}\}$, for which $\forall i \in I$ we have $\sum_{j\in J} G_{i,j} = E_i$ and $\forall j \in J$ we have $\sum_{i\in I} G_{i,j} = F_j$.
(COEX) E and F are *coexistent*, if there exists a POVM $H = \{H_l \geq 0 \mid l \in L,\ \sum_{l\in L} H_l = \mathrm{I}\}$, for which $\operatorname{ran} E \cup \operatorname{ran} F \subseteq \operatorname{ran} H$. (The *range* of a POVM is $\operatorname{ran} E = \{E_{I'} \mid \exists I' \subseteq I \text{ s.t. } E_{I'} = \sum_{i\in I'} E_i\}$.)

We have seen that COM $\Rightarrow$ ND $\Rightarrow$ JM $\Rightarrow$ COEX, in the following exercises we will see that COM $\not\Leftarrow$ ND $\not\Leftarrow$ JM $\not\Leftarrow$ COEX. (After each exercise, think over how it proves the corresponding $\not\Leftarrow$.)

Exercise 19. – Nondisturbance

Let us have $\mathcal{H} = \mathbb{C}^3$, and the two-outcome POVMs $E = \{E_1, E_2\}$ and $F = \{F_1, F_2\}$, and

$$E_1 = \frac{1}{4}\begin{pmatrix} 2 & 0 & -\sqrt{2} \\ 0 & 4 & 0 \\ -\sqrt{2} & 0 & 3 \end{pmatrix}, \qquad F_1 = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Calculate E_2 and F_2.
(b) Do E and F commute?
Let us have the instrument $\{\Phi_i\}$ in the form $\Phi_1(\varrho) = \sum_{k=1}^4 K_{1,k}\varrho K_{1,k}^\dagger$ and $\Phi_2(\varrho) = K_2\varrho K_2^\dagger$ with

$$K_{1,1} = \frac{1}{2}\begin{pmatrix} \sqrt{2} & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ K_{1,2} = \frac{1}{10}\begin{pmatrix} 0 & 0 & 1 \\ 0 & -\sqrt{10} & 2\sqrt{10} \\ 0 & 0 & 0 \end{pmatrix},\ K_{1,3} = \frac{1}{10}\begin{pmatrix} 0 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(c) Construct the Kraus operators $K_{1,4}$ and K_2 in such a way that $\{\Phi_i\}$ implements E, and it does not disturb F.

What are the conclusions?

Exercise 20. – Joint measurability

We will make use of the following.
Lemma: Let us have two POVMs E and F. Suppose that $F_j \in F$ is proportional to a projection for a given $j \in J$. Then, if an instrument $\{\Phi_{E,i}\}$ implements E, and $\operatorname{tr} \sum_i \Phi_{E,i}(\varrho) F_j = \operatorname{tr} \varrho F_j$ (for that given j), then $[E_i, F_j] = 0$ for all $i \in I$ (for that given j).

Let us have the two-outcome noncommuting POVMs $E = \{E_1, E_2\}$ and $F = \{F_1, F_2\}$, and damp them by T-transforms $E_s = \{E_{1,s}, E_{2,s}\}$, $F_t = \{F_{1,t}, F_{2,t}\}$ as

$$\begin{aligned} E_{1,s} &= sE_1 + (1-s)E_2, & E_{2,s} &= (1-s)E_1 + sE_2, & 0 &\leq s \leq 1, \\ F_{1,t} &= tF_1 + (1-t)F_2, & F_{2,t} &= (1-t)F_1 + tF_2, & 0 &\leq t \leq 1. \end{aligned}$$

First step:
Let us form $G = \{G_{1,1}, G_{1,2}, G_{2,1}, G_{2,2}\}$ as $G_{1,1} = a\mathrm{I}$, $G_{1,2} = bE_1$, $G_{2,1} = cF_1$.
(a) Express a, b and c by t such that $G_{1,1} + G_{1,2} = E_{1,t}$, $G_{2,1} + G_{2,2} = E_{2,t}$, $G_{1,1} + G_{2,1} = F_{1,t}$, $G_{1,2} + G_{2,2} = F_{2,t}$.
(b) For which values of t is this a POVM?
Second step:
(c) Calculate the ranges of s and t for which E_s and F_t do not commute.
(d) Prove that E_s and F_t are not jointly measurable for suitable values of s and t. (Hint: prove by contradiction, and use the Lemma above.)
What are the conclusions?
For extra point:
(e) Prove also the Lemma above.

Exercise 21. – Coexistence

Let us have $\mathcal{H} = \mathbb{C}^3$, and the POVMs $E = \{E_1, E_2, E_3\}$ and $F = \{F_1, F_2\}$, and

$$E_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad E_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad F_1 = \frac{1}{6}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

(a) Calculate E_3 and F_2.
(b) Find a POVM H of size $|H| = 5$, for which $\operatorname{ran} E \cup \operatorname{ran} F \subseteq \operatorname{ran} H$.
(c) Show that E and F are not jointly measurable (hint: use proof by contradiction).
What are the conclusions?