Exercise sheet for the course *Quantum entanglement* (Wigner RCP, 2023 spring) for the interested attendants. There is no submission, although we may consult; there is no deadline, although it is helpful to consider the exercises alongside the lectures, as they deepen and illustrate the material.

This document is planned to be updated week-by-week, following the topics of the lectures. This version is [June 21, 2023]. Exercises for previous or extra topics can also be inserted. Please report errors or misprints at szalay.szilard@wigner.hu.

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1 Linear algebra

We have the finite dimensional Hilbert space $\mathcal{H}$, $\dim(\mathcal{H}) =: d$, and we use the Dirac notation. An orthonormal set of vectors spanning $\mathcal{H}$ is called *basis,* e.g., $\{|\alpha_i\rangle \in \mathcal{H} \mid i = 1, 2, \dots, d\}$, $\langle\alpha_i|\alpha_j\rangle = \delta_{i,j}$.

1.1 Trace map

The *trace map* $\mathrm{Tr} : \mathrm{Lin}(\mathcal{H}) \to \mathbb{C}$ is linear, and given as $\mathrm{Tr}(|\alpha\rangle\langle\beta|) = \langle\beta|\alpha\rangle$ on elementary operators $|\alpha\rangle\langle\beta|$.
• Show the cyclicity of the trace, $\mathrm{Tr}(BCA) = \mathrm{Tr}(ABC)$ for all $A \in \mathrm{Lin}(\mathcal{H}'', \mathcal{H})$, $B \in \mathrm{Lin}(\mathcal{H}', \mathcal{H}'')$ and $C \in \mathrm{Lin}(\mathcal{H}, \mathcal{H}')$.
• Show that the trace of an operator equals the usual trace of its matrix.

1.2 Adjoint

The *adjoint map* $\dagger : \mathrm{Lin}(\mathcal{H}, \mathcal{H}') \to \mathrm{Lin}(\mathcal{H}', \mathcal{H})$ is antilinear, and given as $(|\alpha\rangle\langle\beta|)^\dagger = |\beta\rangle\langle\alpha|$ on elementary operators $|\alpha\rangle\langle\beta| \in \mathrm{Lin}(\mathcal{H}, \mathcal{H}')$, that is, $|\alpha\rangle \in \mathcal{H}'$, $|\beta\rangle \in \mathcal{H}$.
• Show that $\langle\alpha|A\beta\rangle = \langle(A^\dagger\alpha)|\beta\rangle$ for all $|\alpha\rangle \in \mathcal{H}'$ and $|\beta\rangle \in \mathcal{H}$.
• Show that the matrix of the adjoint of an operator equals the usual adjoint of its matrix (that is, flipping the matrix to its main diagonal and taking the complex conjugate).

1.3 Hilbert-Schmidt space

The *Hilbert-Schmidt inner product* is given as $(A|B) := \mathrm{Tr}(A^\dagger B)$ for $A, B \in \mathrm{Lin}(\mathcal{H}, \mathcal{H}')$.
• Show that it is indeed an inner product.
(Then $\mathrm{Lin}(\mathcal{H}, \mathcal{H}')$ with this inner product is also a Hilbert space, called the *Hilbert-Schmidt space.*)

1.4 Operators

• Show that for all operators $A \in \mathrm{Lin}(\mathcal{H}, \mathcal{H}')$, there exist scalars $s_i \geq 0$ and bases $\{|\alpha_i\rangle \in \mathcal{H}'\}$, $\{|\beta_i\rangle \in \mathcal{H}\}$ by which

$$A = \sum_{i=1}^{\min\{d,d'\}} s_i |\alpha_i\rangle\langle\beta_i|. \qquad \text{eq:SVD} \quad (1)$$

(This is called *singular value decomposition*, or *singular decomposition*, the $\{s_i\}$ *singular values*, and $\{|\alpha_i\rangle\}$ and $\{|\beta_i\rangle\}$ *left and right singular vectors*.) *(Hint: a simple proof is based on the diagonalization of normal operators, which is the following exercise.)*

Normal operators are the operators $A \in \mathrm{Lin}(\mathcal{H})$ obeying $AA^\dagger = A^\dagger A$.

- Show that A is normal if and only if there exist scalars $a_i \in \mathbb{C}$ and basis $\{|\alpha_i\rangle \in \mathcal{H}\}$ by which

$$A = \sum_{i=1}^{d} a_i |\alpha_i\rangle\langle\alpha_i|. \quad \text{eq:NormalD} \tag{2}$$

(This is called *eigendecomposition*, the $\{a_i\}$ *eigenvalues*, and $\{|\alpha_i\rangle\}$ *eigenvectors*. It is also said that normal operators are *diagonalizable*, and we say that (2) is *diagonal* in the basis $\{|\alpha_i\rangle\}$, since its matrix elements in this basis $\langle\alpha_i|A|\alpha_j\rangle$ form a diagonal matrix $a_i\delta_{ij}$.)

Self-adjoint operators are the operators $A \in \mathrm{Lin}(\mathcal{H})$ obeying $A^\dagger = A$.

- Show that A is self-adjoint if and only if $a_i \in \mathbb{R}$ in the decomposition (2).

Positive (semidefinite) operators are the operators $A \in \mathrm{Lin}(\mathcal{H})$ obeying $\langle\psi|A|\psi\rangle \geq 0$ for all $|\psi\rangle \in \mathcal{H}$. This is denoted as $A \geq 0$.

- Show that positive operators form a pointed (convex) cone.
- Show that A is positive if and only if there exists $B \in \mathrm{Lin}(\mathcal{H}, \mathcal{H}')$ by which $A = B^\dagger B$.
- Show that A is positive if and only if $(A|B) \geq 0$ for all $0 \leq B \in \mathrm{Lin}(\mathcal{H})$.

(That is, the "angle" of the cone of positive operators at the point 0 is at most $\pi/2$.)

- Show that A is positive if and only if $a_i \geq 0$ in the decomposition (2).

Unitary operators are the invertible operators $U \in \mathrm{Lin}(\mathcal{H})$ obeying $U^{-1} = U^\dagger$.

- Show that U is unitary if and only if $\langle U\psi|U\phi\rangle = \langle\psi|\phi\rangle$ for all $|\psi\rangle, |\phi\rangle \in \mathcal{H}$.
- Show that U is unitary if and only if $|a_i| = 1$ in the decomposition (2).

(These operators form the unitary group of $\mathcal{H}$, which is a Lie group, denoted as $\mathrm{U}(\mathcal{H})$.)

Projectors are the self-adjoint operators $P \in \mathrm{Lin}(\mathcal{H})$ obeying $P^2 = P$ (idempotent).

- Show that P is a projector if and only if $a_i = \{0, 1\}$ in the decomposition (2).

We have now the important types of operators over finite dimensional Hilbert spaces.

- What is the relation among these?

1.5 Basis change

Let us have two bases $\{|\alpha_i\rangle \in \mathcal{H}\}$, $\{|\beta_i\rangle \in \mathcal{H}\}$.

- Write the operator mapping $|\alpha_i\rangle \mapsto |\beta_i\rangle$.
- Write its matrixelements in both bases. How to think of these?
- Which kind of operator is this? (From the previous exercise.)

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2 Convexity

Here we consider convexity in finite dimensional vector spaces.

2.1 Cones

A vector space is closed under finite linear combination. Inside that, a cone is a set closed under finite linear combination with *nonnegative real* numbers.
• Show that d-tuples of nonnegative numbers form a cone in $\mathbb{R}^d$.
• Show that positive semidefinite operators over a d-dimensional Hilbert space form a cone in the $\mathbb{R}$-linear space of the self-adjoint operators of the Hilbert space.

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3 States and observables

During the course we consider "discrete finite systems", that is, physical systems on which measurements can result in a finite number of different outcomes.

3.1 Classical states

First, consider classical(ly behaving) systems. The set of *classical discrete probability densities (classical states)* is

eq:Delta

$$\Delta := \Big\{ \boldsymbol{p} \in \mathbb{C}^d \;\Big|\; \boldsymbol{p}^* = \boldsymbol{p}, \boldsymbol{p} \geq 0, \|\boldsymbol{p}\|_1 = \sum_i |p_i| = 1 \Big\}. \tag{3}$$

(Complex conjugation and relation is understood elementwisely.) The set of *pure* states is

eq:Pi

$$\Pi := \Big\{ \boldsymbol{\delta}^j \in \mathbb{C}^d \;\Big|\; j = 1, 2, \dots, d; (\boldsymbol{\delta}^j)_k = \delta^j_k \Big\}. \tag{4}$$

The *mixed* states are the others.
• Show that $\Pi \subset \Delta$, $\Delta = \mathrm{Conv}(\Pi)$, and $\Pi = \mathrm{Extr}(\Delta)$.
• Show that the extremal convex decomposition of a state is unique (that is, Δ is a simplex).
• Show that $\boldsymbol{p} \in \Pi$ if and only if $\boldsymbol{p}^2 = \boldsymbol{p}$. (Taking the power is understood elementwisely.)

3.2 Quantum states

...I know this was easy, let's see the quantum case. The set of *the density operators (states) of discrete finite quantum systems* is

eq:D

$$\mathcal{D} := \Big\{ \rho \in \mathrm{Lin}(\mathcal{H}) \;\Big|\; \rho^\dagger = \rho, \rho \geq 0, \|\rho\|_1 = \mathrm{Tr}(\rho) = 1 \Big\}. \tag{5}$$

The set of *pure* states is

eq:P

$$\mathcal{P} := \Big\{ \pi \in \mathrm{Lin}(\mathcal{H}) \;\Big|\; \pi = |\psi\rangle\langle\psi|, |\psi\rangle \in \mathcal{H}, \|\psi\| = 1 \Big\}. \tag{6}$$

The *mixed* states are the others.
• Show that $\mathcal{P} \subset \mathcal{D}$, $\mathcal{D} = \mathrm{Conv}(\mathcal{P})$, and $\mathcal{P} = \mathrm{Extr}(\mathcal{D})$.

• Which ones of the density operators given by the following matrices represent pure states?

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}, \qquad \begin{bmatrix} 3/4 & 0 \\ 0 & 1/4 \end{bmatrix}, \qquad \begin{bmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{bmatrix}, \qquad \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}.$$

• Show that the extremal convex decomposition of a quantum state is *not* unique (that is, $\mathcal{D}$ is not a simplex). That is, taking a state $\rho \in \mathcal{D}$, we may have many different ways of writing it as $\rho = \sum_{j=1}^{m} w_j |\psi^j\rangle\langle\psi^j|$ (where $\boldsymbol{w} \in \Delta_{m-1}$ and $\|\psi_j\|^2 = 1$). Show some examples, by writing at least two different decompositions of the density operators given by the matrices above. (Have you succeeded?)
• Show that a density operator $\rho \in \mathcal{D}$ is pure if and only if $\rho^2 = \rho$.
• Show that a density operator $\rho \in \mathcal{D}$ is pure if and only if $\mathrm{Tr}(\rho^2) = 1$.

3.3 Qubits and roations

For $\dim(\mathcal{H}) = 2$, the systems are called *qubits*. Let us have the Pauli operators $\sigma_j \in \mathrm{Lin}(\mathcal{H})$ for $j = 1, 2, 3$, self adjoint and traceless, obeying the product rule

eq:Pauli

$$\sigma_j \sigma_k = \delta_{jk} I + i \sum_{l=1}^{3} \epsilon_{jkl} \sigma_l, \tag{7}$$

leading to

eq:Pauliprod

$$(\boldsymbol{x\sigma})(\boldsymbol{y\sigma}) = (\boldsymbol{xy}) I + i(\boldsymbol{x} \times \boldsymbol{y})\boldsymbol{\sigma}, \tag{8}$$

for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^3$, with the notation $\boldsymbol{x\sigma} = x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3$.
• Show that the operator $A = A_0 I + \boldsymbol{A\sigma}$ for $A_0 \in \mathbb{C}$ and $\boldsymbol{A} \in \mathbb{C}^3$ is normal if and only if $\boldsymbol{A} \times \boldsymbol{A}^* = \boldsymbol{0}$, and it is self adjoint if and only if $A_0 \in \mathbb{R}$ and $\boldsymbol{A} \in \mathbb{R}^3$.
• Show/recall that the eigenvalues of $\boldsymbol{x\sigma}$ are $\pm\sqrt{\boldsymbol{xx}}$, and the corresponding eigenprojectors are $|\xi^\pm\rangle\langle\xi^\pm| = \frac{1}{2}(I \pm \frac{\boldsymbol{x\sigma}}{\sqrt{\boldsymbol{xx}}})$.
• Show/recall that $\rho \in \mathcal{D}$ (ρ is a quantum state) if and only if it is of the form

eq:Bloch

$$\varrho = \frac{1}{2}\big(I + \boldsymbol{r\sigma}\big), \qquad \text{where } \boldsymbol{r} \in \mathbb{R}^3,\ \|\boldsymbol{r}\| \leq 1, \tag{9}$$

and $\rho \in \mathcal{P}$ (ρ is a pure quantum state) if and only if $\|\boldsymbol{r}\| = 1$ above. (The vectors $\boldsymbol{r}$ in (9) describing the states of qubits are called *Bloch vectors*, they form the *Bloch ball* B^3, and, in the case of pure states, the *Bloch sphere* S^2.)
• Write the eigenvalues and eigenprojectors of ϱ.
(These can be done without the concrete matrices, only the product rule (8) of the Pauli operators is needed.)

Let us have the unit vector $\hat{\boldsymbol{u}} \in \mathbb{R}^3$ ($\|\hat{\boldsymbol{u}}\|^2 = 1$), the angle $\gamma \in \mathbb{R}$, and the operator

eq:U

$$U_{\hat{\boldsymbol{u}}}(\gamma) := \mathrm{e}^{-i\frac{\gamma}{2}\hat{\boldsymbol{u}}\boldsymbol{\sigma}}. \tag{10}$$

- Carry out the exponentialization, that is, find the parameters $s \in \mathbb{C}$ and $\boldsymbol{s} \in \mathbb{C}^3$ for which $U_{\hat{\boldsymbol{u}}}(\gamma) = sI + \boldsymbol{s\sigma}$.
- Find the determinant of $U_{\hat{\boldsymbol{u}}}(\gamma)$.
- Find the inverse of $U_{\hat{\boldsymbol{u}}}(\gamma)$.

(These can be done without the concrete matrices, only the product rule (8) of the Pauli operators is needed.)

Let us have the Σ_i operators for $i = 1, 2, 3$, acting on $\mathbb{R}^3$ as $(\boldsymbol{x\Sigma})\boldsymbol{y} = i(\boldsymbol{x} \times \boldsymbol{y})$, with the notation $\boldsymbol{x\Sigma} = x_1\Sigma_1 + x_2\Sigma_2 + x_3\Sigma_3$. Let us have the unit vector $\hat{\boldsymbol{u}} \in \mathbb{R}^3$ ($\|\hat{\boldsymbol{u}}\|^2 = 1$), the angle $\gamma \in \mathbb{R}$, and the operator

$$R_{\hat{\boldsymbol{u}}}(\gamma) = \mathrm{e}^{-i\gamma\hat{\boldsymbol{u}}\boldsymbol{\Sigma}}. \tag{11}$$

- Carry out the exponentialization.
- Find the determinant of $R_{\hat{\boldsymbol{u}}}(\gamma)$.
- Find the inverse of $R_{\hat{\boldsymbol{u}}}(\gamma)$.
- What is the meaning of $R_{\hat{\boldsymbol{u}}}(\gamma)$?

(These can be done without the concrete matrices, only some known identities of the vectorial product $\times$ are needed.)

Now we connect the two.

- Show that the adjoint action of $U_{\hat{\boldsymbol{u}}}(\gamma)$ in $\mathrm{Lin}(\mathcal{H})$ is the same as the action of $R_{\hat{\boldsymbol{u}}}(\gamma)$ on the Bloch sphere. That is,

$$\begin{aligned} \rho \quad &\longmapsto \quad \rho' = U_{\hat{\boldsymbol{u}}}(\gamma)\rho U_{\hat{\boldsymbol{u}}}(\gamma)^\dagger, \\ \boldsymbol{r} \quad &\longmapsto \quad \boldsymbol{r}' = R_{\hat{\boldsymbol{u}}}(\gamma)\boldsymbol{r}. \end{aligned}$$

Note that $\frac{1}{2}\sigma_i$ and Σ_i are the 2 and 3 dimensional representations of the same Lie algebra. Both obey the commutation relation

$$[J_j, J_k] = i\sum_{l=1}^{3} \epsilon_{jkl} J_l,$$

defining the Lie algebra $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$, and $\frac{1}{2}\sigma_i$ and Σ_i are the so called spin-$\frac{1}{2}$ and spin-1 representations. Then the exponential map (see equations (10) and (11)) leads to $U_{\hat{\boldsymbol{u}}}(\gamma)$ and $R_{\hat{\boldsymbol{u}}}(\gamma)$, which give the spin-$\frac{1}{2}$ and spin-1 representations of the Lie group SU(2). (The 2 dimensional representation is the defining representation.) The spin-1 representation is also the (3 dimensional, defining) representation of SO(3), describing the rotations of the three dimensional space. The Lie group SU(2) is the double covering of SO(3), which can also be seen from $U_{\hat{\boldsymbol{u}}}(\gamma)\boldsymbol{x\sigma}U_{\hat{\boldsymbol{u}}}(\gamma)^\dagger = (R_{\hat{\boldsymbol{u}}}(\gamma)\boldsymbol{x})\boldsymbol{\sigma}$, since $U_{\hat{\boldsymbol{u}}}(2\pi) = -I$, and $\pm U_{\hat{\boldsymbol{u}}}(\gamma)$ lead to the same $R_{\hat{\boldsymbol{u}}}(\gamma)$.

3.4 Qubits and matrices

The matrices of the Pauli operators are

$$\sigma_1 : \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 : \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 : \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and for $\boldsymbol{x} \in \mathbb{R}^3$ we use the notation $\boldsymbol{x\sigma} = x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3$. The operators corresponding to the three spin directions are $S_i = \frac{\hbar}{2}\sigma_i$. Then the spin operator corresponding to the direction $\hat{\boldsymbol{v}} = \big(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)\big) \in \mathbb{R}^3$ (so $\|\hat{\boldsymbol{v}}\|^2 = 1$) is $S_{\hat{\boldsymbol{v}}} := \hat{\boldsymbol{v}}\boldsymbol{S} = \frac{\hbar}{2}\hat{\boldsymbol{v}}\boldsymbol{\sigma}$.

- Calculate the matrix of $S_{\hat{\boldsymbol{v}}}$, its eigenvalues, the matrices of its eigenprojectors, and the 2-tuples of its eigenvectors.
- Write these out also for the concrete cases of the six main coordinate directions $\boldsymbol{v} = (+1,0,0)$, $(-1,0,0)$, $(0,+1,0)$, $(0,-1,0)$, $(0,0,+1)$ and $(0,0,-1)$.
- Calculate the matrix of ρ given in (9), its eigenvalues, the matrices of its eigenprojectors, and the 2-tuples of its eigenvectors.
- Write these out also for the concrete cases of the six main coordinate directions $\boldsymbol{v} = (+1,0,0)$, $(-1,0,0)$, $(0,+1,0)$, $(0,-1,0)$, $(0,0,+1)$ and $(0,0,-1)$.

The so called *Hadamard transform* is the adjoint action of the operator of the matrix

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \in \mathrm{U}(2). \tag{12}$$

(The concrete matrix form is coming from the discrete Fourier transformation for $d = 2$.)

- Calculate the determinant of this.
- Calculate the inverse of this.
- How to imagine this, how does this act on the Bloch ball? That is, for a ρ of the form given in (9),

$$\begin{array}{ccc} \varrho & \longmapsto & H\varrho H^\dagger, \\ \boldsymbol{r} & \longmapsto & \boldsymbol{r}' =? \end{array}$$

- Can this transformation be given as the adjoint action of operators of the form (10)?

The so called *flip operation* on the state of a qubit is complex conjugation of its matrix (in the usual σ_3 eigenbasis) followed by the adjoint action of σ_2. (This is an anti-unitary transformation.)

- How to imagine this, how does this act on the Bloch ball? That is, for a ρ of the form given in (9),

$$\begin{array}{ccc} \varrho & \longmapsto & \tilde{\varrho} = \sigma_2\varrho^*\sigma_2^\dagger, \\ \boldsymbol{r} & \longmapsto & \boldsymbol{r}' =? \end{array}$$

(Star denotes the elementwise complex conjugation of the matrix.)

- Calculate the inverse of this.
- Can this transformation be given as the adjoint action of operators of the form (10)?
- What if we drop the complex conjugation?

3.5 Qutrits and matrices

It is known in general that the d^2-1 (self-adjoint, traceless) generators of the Lie group SU(d) can be choosen to obey the product rule

eq:SUd

$$\sigma_j\sigma_k = \frac{2}{d}\delta_{jk}I + \sum_{l=1}^{d^2-1}(d_{jkl}+if_{jkl})\sigma_l, \tag{13}$$

where the coefficients d_{jkl} and f_{jkl} are real, and d_{jkl} is completely symmetric and f_{jkl} is completely antisymmetric with respect to the permutations of the indices. (For the special case of SU(2), we had $d_{jkl}=0$ and $f_{jkl}=\epsilon_{jkl}$, see equation (7).)

In the case of qutris, we have $\dim(\mathcal{H}) = d = 3$, and we can use the basis $\{I, \sigma_j \mid j = 1,2,\dots 8\}$ in the real vector space of self-adjoint linear operators, where I is the identity operator, and the operators σ_j are the generators of SU(3), given by the so called Gell-Mann matrices. The usual representations of these are

$$\sigma_1: \begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}, \qquad \sigma_2: \begin{bmatrix}0&-i&0\\i&0&0\\0&0&0\end{bmatrix}, \qquad \sigma_3: \begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix},$$

$$\sigma_4: \begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}, \qquad \sigma_5: \begin{bmatrix}0&0&-i\\0&0&0\\i&0&0\end{bmatrix},$$

$$\sigma_6: \begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}, \qquad \sigma_7: \begin{bmatrix}0&0&0\\0&0&-i\\0&i&0\end{bmatrix}, \qquad \sigma_8: \frac{1}{\sqrt{3}}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-2\end{bmatrix}.$$

Then the state of a qutrit can be written by the density operator

$$\rho = \frac{1}{3}(I + \boldsymbol{r}\boldsymbol{\sigma}),$$

where $\boldsymbol{r} \in \mathbb{R}^8$ is the *(generalized) Bloch vector*. Contrary to the case of qubits, now we cannot describe the compact closed convex set (convex body) $C \subset \mathbb{R}^8$ in terms of Bloch vectors, inside which the vectors $\boldsymbol{r}$ lead to a state, that is, a positive semidefinite operator.

• Write out the matrix of ρ.
• Consider particular cases, that is, fix some coordinates as
$(0,0,\dots r_j,\dots,0)$ (only one nonzero),
$(0,0,r_3,0,0,0,0,r_8)$ (diagonal),
$(r_1,r_2,r_3,0,0,0,0,0)$ (what is this?),
$(0,0,r_3,r_4,r_5,0,0,\sqrt{3}r_3)$ (and this?),
$(r_1,0,0,r_4,0,r_6,0,0)$ (a bit more interesting).
What are the possible ranges of the parameters in $\boldsymbol{r}$ in these cases?
• Voluntary task: check if the Gell-Mann matrices above obey equation (13). If they do, what are the values of the coefficients d_{jkl} and f_{jkl}? If they do not, what are the

coefficients of σ_l in the product $\sigma_j \sigma_k$? (This may be tedious by pen and paper, feel free to implement it in arbitrary computer algebra system.)

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4 Events, logic

This was presented later in the semester for didactic reasons, but should come here logically.

4.1 Classical events, Boolean lattice

Let us have the usual representation of the classical event algebra $\Sigma = \{0,1\}^d$ (the set of d-tuples of 0s and 1s), with the elements $\mathbf{0} = (0,0,\dots,0) \in \Sigma$ and $\mathbf{1} = (1,1,\dots,1) \in \Sigma$, and the operations

$$\begin{aligned} \boldsymbol{\alpha} \wedge \boldsymbol{\beta} &:= \boldsymbol{\alpha} \cdot \boldsymbol{\beta}, \\ \boldsymbol{\alpha} \vee \boldsymbol{\beta} &:= \boldsymbol{\alpha} + \boldsymbol{\beta} - \boldsymbol{\alpha} \cdot \boldsymbol{\beta}, \\ \overline{\boldsymbol{\alpha}} &:= \mathbf{1} - \boldsymbol{\alpha}, \end{aligned}$$

for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Sigma$. (Here not only $+$ but also $\cdot$ is meant elementwisely.)
- Check that Σ is closed with respect to the three operations above.
- Check the following properties:

$$\begin{aligned} \boldsymbol{\alpha} \wedge \boldsymbol{\beta} &= \boldsymbol{\beta} \wedge \boldsymbol{\alpha}, & \boldsymbol{\alpha} \vee \boldsymbol{\beta} &= \boldsymbol{\beta} \vee \boldsymbol{\alpha}, \\ (\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) \wedge \boldsymbol{\gamma} &= \boldsymbol{\alpha} \wedge (\boldsymbol{\beta} \wedge \boldsymbol{\gamma}), & (\boldsymbol{\alpha} \vee \boldsymbol{\beta}) \vee \boldsymbol{\gamma} &= \boldsymbol{\alpha} \vee (\boldsymbol{\beta} \vee \boldsymbol{\gamma}), \\ \boldsymbol{\alpha} \wedge (\boldsymbol{\alpha} \vee \boldsymbol{\beta}) &= \boldsymbol{\alpha}, & \boldsymbol{\alpha} \vee (\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) &= \boldsymbol{\alpha}, \\ \boldsymbol{\alpha} \wedge (\boldsymbol{\beta} \vee \boldsymbol{\gamma}) &= (\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) \vee (\boldsymbol{\alpha} \wedge \boldsymbol{\gamma}), & \boldsymbol{\alpha} \vee (\boldsymbol{\beta} \wedge \boldsymbol{\gamma}) &= (\boldsymbol{\alpha} \vee \boldsymbol{\beta}) \wedge (\boldsymbol{\alpha} \vee \boldsymbol{\gamma}), \\ \boldsymbol{\alpha} \wedge \mathbf{0} &= \mathbf{0}, & \boldsymbol{\alpha} \vee \mathbf{0} &= \boldsymbol{\alpha}, \\ \boldsymbol{\alpha} \wedge \mathbf{1} &= \boldsymbol{\alpha}, & \boldsymbol{\alpha} \vee \mathbf{1} &= \mathbf{1}, \\ \boldsymbol{\alpha} \wedge \overline{\boldsymbol{\alpha}} &= \mathbf{0}, & \boldsymbol{\alpha} \vee \overline{\boldsymbol{\alpha}} &= \mathbf{1}, \end{aligned}$$

for all $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \Sigma$. That is, Σ is a Boolean lattice. (Some of these are trivial, kept for the sake of completeness.)

Let us have the partial order $\leq$ defined as $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$ if and only if $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = \boldsymbol{\alpha}$.
- Write the events $\boldsymbol{\alpha} \in \Sigma$ for which $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$ holds for a given $\boldsymbol{\beta} \in \Sigma$.
- Write the events $\boldsymbol{\beta} \in \Sigma$ for which $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$ holds for a given $\boldsymbol{\alpha} \in \Sigma$.

This gives us a hint about the meaning of $\leq$ in Σ.
- What is that?

4.2 Classical probability measures

A classical probability measure is a function $p : \Sigma \to [0,1]$, which is normalized ($p(\mathbf{1}) = 1$) and (σ-)additive (if $\{\boldsymbol{\alpha}_i \in \Sigma\}$ is such that $\boldsymbol{\alpha}_i \leq \overline{\boldsymbol{\alpha}_{j\neq i}}$, then $p(\bigvee_i \boldsymbol{\alpha}_i) = \sum_i p(\boldsymbol{\alpha}_i)$). For all

classical states $\boldsymbol{p} \in \Delta$, the map $\boldsymbol{\alpha} \mapsto p_{\boldsymbol{p}} = (\boldsymbol{p}|\boldsymbol{\alpha}) = \sum_i p_i \alpha_i$ gives a classical probability measure.
- Check this for the finite dimensional case.
- Check that $p(\boldsymbol{\alpha} \vee \boldsymbol{\beta}) = p(\boldsymbol{\alpha}) + p(\boldsymbol{\beta}) - p(\boldsymbol{\alpha} \wedge \boldsymbol{\beta})$.
- How to imagine this?
- Show that if $p(\boldsymbol{\alpha}) = p(\boldsymbol{\beta}) = 1$ then $p(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) = 1$.

4.3 Quantum events, Hilbert lattice

Let us have $\mathcal{L}$, the set of projectors to the subspaces of the d dimensional Hilbert space $\mathcal{H}$, with the elements $0 \in \mathcal{L}$ and $I \in \mathcal{L}$ (projecting to the subspaces $\emptyset \subseteq \mathcal{H}$ and $\mathcal{H} \subseteq \mathcal{H}$) and

$$\begin{aligned} P \wedge Q &:= \mathrm{Proj}(\mathrm{Ran}(P) \cap \mathrm{Ran}(Q)), \\ P \vee Q &:= \mathrm{Proj}(\mathrm{Span}(\mathrm{Ran}(P) \cup \mathrm{Ran}(Q))), \\ \overline{P} &:= \mathrm{Proj}(\mathrm{Ran}(P)^{\perp}), \end{aligned}$$

for all $P, Q \in \mathcal{L}$. (On the right-hand side, $\mathrm{Ran}(P)$ is the range of the projector P, which is the subspace P projects onto, and $\mathrm{Proj}(\mathcal{K})$ is the projector of range $\mathcal{K}$.)
$\mathcal{L}$ is obviously closed with respect to the three operations above.
- Show that $P^{\perp} = I - P$.
- Show that if $PQ = QP$ then $P \wedge Q = PQ$ and $P \vee Q = P + Q - PQ$.
- Check the following properties:

$$\begin{aligned} P \wedge Q &= Q \wedge P, & P \vee Q &= Q \vee P, \\ (P \wedge Q) \wedge R &= P \wedge (Q \wedge R), & (P \vee Q) \vee R &= P \vee (Q \vee R), \\ P \wedge (P \vee Q) &= P, & P \vee (P \wedge Q) &= P, \\ P \wedge 0 &= 0, & P \vee 0 &= P, \\ P \wedge I &= P, & P \vee I &= I, \\ P \wedge \overline{P} &= 0, & P \vee \overline{P} &= I, \\ \overline{P \wedge Q} &= \overline{P} \vee \overline{Q}, & \overline{P \vee Q} &= \overline{P} \wedge \overline{Q}, \end{aligned}$$

for all $P, Q, R \in \mathcal{L}$. That is, $\mathcal{L}$ is an orthocomplemented bounded lattice. (Some of these are trivial, kept for the sake of completeness.)
- Show an example illustrating that $\mathcal{L}$ is not distributive, that is,

$$P \wedge (Q \vee R) \neq (P \wedge Q) \vee (P \wedge R), \qquad P \vee (Q \wedge R) \neq (P \vee Q) \wedge (P \vee R).$$

Let us have the partial order $\leq$ defined as $P \leq Q$ if and only if $P \wedge Q = P$.
- Show that $P \leq Q$ if and only if $\mathrm{Ran}(P) \subseteq \mathrm{Ran}(Q)$.
- Write the events $P \in \mathcal{L}$ for which $P \leq Q$ holds for a given $Q \in \mathcal{L}$.
- Write the events $Q \in \mathcal{L}$ for which $P \leq Q$ holds for a given $P \in \mathcal{L}$.
- Show that $PQ = QP$ if and only if $P \leq Q$ or $Q \leq P$.

4.4 Quantum probability measures

A quantum probability measure is a function $q : \mathcal{L} \to [0,1]$, which is normalized ($q(I) = 1$), ($\sigma$-)additive (if $\{P_i \in \mathcal{L}\}$ is such that $P_i \leq \overline{P_{j\neq i}}$, then $q(\bigvee_i P_i) = \sum_i q(P_i)$), and if $q(P) = q(Q) = 1$ then $q(P \wedge Q) = 1$. For all density operators $\rho \in \mathcal{D}$, the map $P \mapsto q_\rho(P) = (\rho|P) = \mathrm{Tr}(\rho P)$ gives a quantum probability measure.
• Check this for the finite dimensional case.
• Show an example for that

$$q_\rho(P \vee Q) \neq q_\rho(P) + q_\rho(Q) - q_\rho(P \wedge Q).$$

• Why is this disturbing?
• Show that for all $P, Q \in \mathcal{L}$ such that $QP \neq PQ$, one can find a $\rho \in \mathcal{D}$ such that

$$1 < q_\rho(P) + q_\rho(Q) - q_\rho(P \wedge Q).$$

• Why is this disturbing?

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5 Bipartite systems

5.1 Classical bipartite systems

Let us have two classical bivalue observables (two bits) represented as $\boldsymbol{a} = (a^1, a^2) \in \mathbb{C}^2$ and $\boldsymbol{b} = (b^1, b^2) \in \mathbb{C}^2$. The composite state (joint probability distribution) of this two-bit system is $\boldsymbol{p}_{12} \in \Delta_{12} \subset \mathbb{R}^2 \otimes \mathbb{R}^2$, given as

$$\begin{aligned}&(\boldsymbol{p}_{12})_{i,j} = p_{12;i,j}\\&\quad = \mathbb{P}(\text{“outcome } i \text{ happened measuring } \boldsymbol{a} \textit{ and } \text{outcome } j \text{ happened measuring } \boldsymbol{b}\text{”}),\end{aligned}$$

The reduced states (marginal distributions) $\boldsymbol{p}_1 \in \Delta_1 \subset \mathbb{R}^2$ and $\boldsymbol{p}_2 \in \Delta_2 \subset \mathbb{R}^2$ are

$$\begin{aligned}(\boldsymbol{p}_1)_i &= p_{1;i} = \mathbb{P}(\text{“outcome } i \text{ happened measuring } \boldsymbol{a}\text{”}),\\(\boldsymbol{p}_2)_j &= p_{2;j} = \mathbb{P}(\text{“outcome } j \text{ happened measuring } \boldsymbol{b}\text{”}).\end{aligned}$$

Recall, how to write $p_{1;i}$ and $p_{2;j}$ in terms of $p_{12;i,j}$. The uncorrelated states are the elementary tensors, $\boldsymbol{p}_{12} = \boldsymbol{p}_1 \otimes \boldsymbol{p}_2$, these form a two-dimensional quadratic submanifold inside Δ_{12}.
• Write out the system of equations describing this submanifold (in terms of $p_{12;i,j}$) and find a convenient parametrization. How can this be pictured inside the simplex Δ_{12}?
Joint states $\boldsymbol{p}_{12}$ having fixed marginals $\boldsymbol{p}_1$ also form a two-dimensional submanifold inside Δ_{12}.
• Write out the system of equations describing this submanifold (in terms of $p_{12;i,j}$) and find a convenient parametrization. How can this be pictured inside the simplex Δ_{12}?

5.2 Quantum bipartite systems

Let us have the Hilbert space of a joint quantum system as $\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Then the partial trace maps are the *linear* maps given as

$$\begin{aligned} \mathrm{Tr}_1 &: \quad \mathrm{Lin}(\mathcal{H}_{12}) \longrightarrow \mathrm{Lin}(\mathcal{H}_2), \\ \mathrm{Tr}_2 &: \quad \mathrm{Lin}(\mathcal{H}_{12}) \longrightarrow \mathrm{Lin}(\mathcal{H}_1), \end{aligned}$$

given on elementary tensor operators as

$$\mathrm{Tr}_1(A \otimes B) = \mathrm{Tr}(A)B, \qquad \mathrm{Tr}_2(A \otimes B) = A\,\mathrm{Tr}(B),$$

with the usual trace map.

• Write the effect of the partial trace on matrix elements, that is, if

$$R = \sum_{i,j,i',j'=1}^{d_1,d_2,d_1,d_2} R^{i}{}_{i'}{}^{j}{}_{j'} |i\rangle\langle i'| \otimes |j\rangle\langle j'|,$$

then write the matirx elements $\mathrm{Tr}_1(R)^j{}_{j'}$ and $\mathrm{Tr}_2(R)^i{}_{i'}$ of the operators

$$\mathrm{Tr}_1(R) = \sum_{j,j'=1}^{d_2,d_2} \mathrm{Tr}_1(R)^j{}_{j'} |j\rangle\langle j'|, \qquad \mathrm{Tr}_2(R) = \sum_{i,i'=1}^{d_1,d_1} \mathrm{Tr}_2(R)^i{}_{i'} |i\rangle\langle i'|.$$

(We use the shorthand notation $|i\rangle := |\phi_i\rangle$ for the elements of a *fixed* basis.)

• How to picture this with block matrices?

5.3 2-qubit canonical form

Let us have $|\psi_{12}\rangle \in \mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$, $\dim(\mathcal{H}_1) = \dim(\mathcal{H}_2) = 2$, and the basis $|\phi_{1,i}\rangle \otimes |\phi_{2,j}\rangle \equiv |\phi_{12,ij}\rangle =: |ij\rangle$ using the usual shorthand notation, so $|\psi_{12}\rangle = \psi^{00}|00\rangle + \psi^{01}|01\rangle + \psi^{10}|10\rangle + \psi^{11}|11\rangle \in \mathcal{H}_{12}$ in general. Let us have

$$|\psi_{12}\rangle = \cos(\alpha)|00\rangle + \sin(\alpha)|11\rangle, \qquad 0 \le \alpha \le \pi/2.$$

(α is called Schmidt angle.)

• Determine the Schmidt coefficients.

• Write out the matrix of the pure state density operator $\pi_{12} = |\psi_{12}\rangle\langle\psi_{12}|$ expressed in the basis above.

• Write out the matrix of the reduced density operators $\rho_1 = \mathrm{Tr}_2(\pi_{12})$, $\rho_2 = \mathrm{Tr}_1(\pi_{12})$.

• Determine the eigenvalues of those. Determine the Schmidt angles leading to pure or strictly mixed states. (What is the original state in those cases?)

Let us have now

$$|\psi_{12}\rangle = x|00\rangle + x|01\rangle + y|10\rangle - y|11\rangle.$$

• Determine the possible values of the coefficients $x, y \in \mathbb{C}$.

• Determine the Schmidt coefficients.

• Write out the matrix of the pure state density operator $\pi_{12} = |\psi_{12}\rangle\langle\psi_{12}|$ expressed in the basis above.
• Write out the matrix of the reduced density operators $\rho_1 = \mathrm{Tr}_2(\pi_{12})$, $\rho_2 = \mathrm{Tr}_1(\pi_{12})$.
• Determine the eigenvalues of those. Determine the values of x, y leading to pure or strictly mixed states. (What is the original state in those cases?)

The *Bell state vectors* of two qubits are given by the state vectors

$$\begin{aligned} |\mathrm{B}_0\rangle &= \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big), & |\mathrm{B}_1\rangle &= \frac{1}{\sqrt{2}}\big(|01\rangle + |10\rangle\big), \\ |\mathrm{B}_2\rangle &= \frac{-i}{\sqrt{2}}\big(|01\rangle - |10\rangle\big), & |\mathrm{B}_3\rangle &= \frac{1}{\sqrt{2}}\big(|00\rangle - |11\rangle\big). \end{aligned} \tag{14}$$

eq:Bells

(They form a basis of the two-qubit Hilbert space, also called *magic basis*.)
• Check that this is indeed a basis, that is, complete orthonormal system in $\mathcal{H}_{12}$.
• Determine the Schmidt coefficients.
• Write out the matrix of the pure state density operators $\pi_{12} = |\mathrm{B}_\mu\rangle\langle\mathrm{B}_\mu|$ expressed in the basis above.
• Write out the matrix of the reduced density operators $\rho_1 = \mathrm{Tr}_2(\pi_{12})$, $\rho_2 = \mathrm{Tr}_1(\pi_{12})$.
• Determine the eigenvalues of those.
• Show that the Bell state vectors are local-unitary equivalent with each other, by checking that $|\mathrm{B}_\mu\rangle = \sigma_\mu \otimes I|\mathrm{B}_0\rangle$ (write out the vectors $|\mathrm{B}_\mu\rangle$ and the matrices $\sigma_\mu \otimes I$).

5.4 Qubit purification

Let $\dim \mathcal{H} = 2$, and let us have the density operator ρ of a qubit, written in the usual Bloch-vector form, see (9). Let us have another system (described by a suitable Hilbert space $\mathcal{H}'$), and write a pure state $\pi \in \mathcal{P}(\mathcal{H} \otimes \mathcal{H}')$ of the joint system, which has the same reduced state as ρ above, $\rho = \mathrm{Tr}_{\mathcal{H}'}(\pi)$. (Such pure states are called *purifications*, or *pure extensions* of the original state.)
• Write all the possible purifications for the case of $\dim(\mathcal{H}') = 2$. This can be done by applying $I \otimes U$ to a particular purification, where $U \in \mathrm{U}(\mathcal{H}')$. (Hint: use the parametrization for the unitary as follows,

$$U = \mathrm{e}^{i\phi/2} \begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix} \in \mathrm{U}(2), \qquad 0 \leq \phi < 2\pi, \quad a, b \in \mathbb{C} \quad \text{és} \quad |a|^2 + |b|^2 = 1.$$

For those who do not know this: check that this is a unitary.)
• Do we get all the possible purifications (for the $\dim(\mathcal{H}') = 2$ case) in this way?
• Can two different unitaries lead to the same purification?

5.5 2-qubit operators, density matrices

Let us have $|\psi_{12}\rangle \in \mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$, $\dim(\mathcal{H}_1) = \dim(\mathcal{H}_2) = 2$, The operators $\{\sigma_0 \equiv I, \sigma_1, \sigma_2, \sigma_3\}$ form a basis for $\mathrm{Lin}(\mathcal{H}_1)$ and $\mathrm{Lin}(\mathcal{H}_2)$. (This led to the Bloch vector description (9) of the qubit state space earlier.) We have the natural tensor product basis

$\{\sigma_\mu \otimes \sigma_\nu \mid \mu,\nu = 0,1,2,3\}$ for $\mathrm{Lin}(\mathcal{H}_{12})$. A two-qubit state can then be written as

$$\rho_{12} = \sum_{iji'j'=0}^{1} \rho^{i}{}_{i'}{}^{j}{}_{j'} |i\rangle\langle i'|\otimes|j\rangle\langle j'| = \sum_{\mu\nu=0}^{3} R^{\mu\nu}\sigma_\mu\otimes\sigma_\nu = \frac{1}{4}\Big[I\otimes I + \boldsymbol{r\sigma}\otimes I + I\otimes\boldsymbol{s\sigma} + \boldsymbol{t\sigma}\otimes\boldsymbol{\sigma}\Big],$$

where $\boldsymbol{r},\boldsymbol{s} \in \mathbb{R}^3$, $\boldsymbol{t} \in \mathbb{R}^3\otimes\mathbb{R}^3$, and we use the shorthand notation $\boldsymbol{r\sigma} = \sum_{i=1}^3 x^i\sigma_i$, $\boldsymbol{t\sigma}\otimes\boldsymbol{\sigma} = \sum_{ij=1}^3 t^{ij}\sigma_i\otimes\sigma_j$. We may think of this as

$$R^{\mu\nu} = \frac{1}{4}\left[\begin{array}{c|c} 1 & \boldsymbol{s} \\ \hline \boldsymbol{r} & \boldsymbol{t} \end{array}\right] \in \mathbb{R}^4\otimes\mathbb{R}^4$$

as well.

- Write the reduced states $\rho_1 = \mathrm{Tr}_2(\rho_{12})$ and $\rho_2 = \mathrm{Tr}_1(\rho_{12})$ for all the three forms above.
- How to write the coefficients $R^{\mu\nu}$ for uncorrelated state ($\rho_{12} = \rho_1\otimes\rho_2$)?

Unfortunately, we cannot formulate a simple condition for the positivity $\rho_{12} \geq 0$ in terms of the coefficients $R^{\mu\nu}$ (or $\boldsymbol{r},\boldsymbol{s},\boldsymbol{t}$) in general. (Recall that this could be given by the length of the Bloch vector in the case of one qubit, led to the Bloch ball.) However, we may consider some particular cases. The *Pauli diagonal states* are those for which $\boldsymbol{r} = \boldsymbol{0}$, $\boldsymbol{s} = \boldsymbol{0}$ and $\boldsymbol{t}$ is diagonal.

- Write the matrix $\rho^{i}{}_{i'}{}^{j}{}_{j'}$ of Pauli-diagonal states.
- Calculate its eigenvalues.
- Give the conditions for $\rho_{12} \geq 0$ in terms of $\boldsymbol{t}$.
- What is the geometry of these in terms of the vector $(t_{11}, t_{22}, t_{33}) \in \mathbb{R}^3$ of prameters?
- Give the conditions in terms of $\boldsymbol{t}$ for a Pauli diagonal state to be pure.

The *Bell diagonal states* are the mixtures of the Bell states (14), $\rho = \sum_{\mu=0}^3 w_\mu |\mathrm{B}_\mu\rangle\langle\mathrm{B}_\mu|$, where $w_\mu \geq 0$, $\sum_{\mu=0}^3 w_\mu = 1$. These are parametrized by the $\boldsymbol{w} \in \Delta \subset \mathbb{R}^4$ weights, taking place in the 3 dimensional simplex Δ.

- Calculate the eigenvalues of these states.
- Calculate also the $R^{\mu\nu}$ coefficients of these states. (For this, calculate the coefficients $R^{\mu\nu}$ of the pure states $\pi_\mu = |\mathrm{B}_\mu\rangle\langle\mathrm{B}_\mu|$.)
- What is then the geometry of the Pauli diagonal states?

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6 State transformations

6.1 Classical channels

We have the classical state space Δ, (3), the linear maps $\boldsymbol{M} : \Delta \to \Delta'$ (positive and sum-preserving) are called stochastic maps (Markov maps, classical channels), or bistochastic in the case when it maps white noise to white noise. These can be represented by stochastic matrices, which are real $d'\times d$ matrices $\boldsymbol{M}$, for which $M_{ij} \geq 0$ (“positive”) and

$\sum_{i=1}^{d'} M_{ij} = 1$ ("sum preserving"), and if moreover bistochastic then $\sum_{j=1}^{d} M_{ij} = d/d'$ ("noise preserving").

The deterministic dynamics $\Delta \to \Delta$ of a system maps pure states onto pure states.
- Write the stochastic matrices $\boldsymbol{M}$ representing such mapping.
- Are they bistochastic?

The dynamics of a closed system is deterministic and also reversible.
- Write the stochastic matrices $\boldsymbol{M}$ representing such mapping.
- Are they bistochastic?
- Are reversible maps always deterministic?

A supplementing system (also called "ancilla", described by the state $\boldsymbol{q} \in \Delta_2$) can be appended to the system in an uncorrelated way, that is, $\boldsymbol{M}_{\boldsymbol{q},2} : \Delta_1 \to \Delta_{12}$, given as $\boldsymbol{M}_{\boldsymbol{q},2}\boldsymbol{p} = \boldsymbol{p} \otimes \boldsymbol{q}$.
- Write the stochastic matrix $\boldsymbol{M}_{\boldsymbol{q},2}$, and also $\boldsymbol{M}_{\boldsymbol{q},1}$, given similarly.
- Is it bistochastic?

A subsystem of a composite system can be discarded, that is, $\boldsymbol{M}_{\mathrm{red}2} : \Delta_{12} \to \Delta_1$, given by the state reduction as $\boldsymbol{M}_{\mathrm{red}2}(\boldsymbol{p}_{12}) = \mathrm{Red}_2(\boldsymbol{p}_{12})$.
- Write the stochastic matrix $\boldsymbol{M}_{\mathrm{red}2}$, and also $\boldsymbol{M}_{\mathrm{red}1}$, given similarly.
- Is it bistochastic?

To a system, described by the state $\boldsymbol{p} \in \Delta$, a system of the same kind, described by the state $\boldsymbol{q} \in \Delta$ can be mixed, that is, $\boldsymbol{M}_{\boldsymbol{q},x} : \Delta \to \Delta$, given as $\boldsymbol{M}_{\boldsymbol{q},x}\boldsymbol{p} = (1-x)\boldsymbol{p} + x\boldsymbol{q}$.
- Write the stochastic matrix $\boldsymbol{M}_{\boldsymbol{q},x}$.
- In which case is it bistochastic?

A system can entirely be replaced with a system, described by the state $\boldsymbol{q} \in \Delta$ (that is, without respect to the original state), that is, $\boldsymbol{M}_{\boldsymbol{q}} : \Delta \to \Delta$, given as $\boldsymbol{M}_{\boldsymbol{q}}\boldsymbol{p} = \boldsymbol{q}$ for all $\boldsymbol{p}$. (This can be considered as a "perparation".)
- Write the stochastic matrix $\boldsymbol{M}_{\boldsymbol{q}}$.
- which case is it bistochastic?

(If you are not fluent enough in using $\otimes$ and the general formalism, you can submit this exercise calculating for bits, $d = 2$.)

6.2 Environmental representation of classical channels

The effect of every stochastic map can be given as the effect of an interacting ancillary system.
- Show this, that is, show that for all $\boldsymbol{M}_1 : \Delta_1 \to \Delta_1$, there exist ancillary system of state $\boldsymbol{q} \in \Delta_2$ and joint reversible dynamics (interaction) $\boldsymbol{M}_{12} : \Delta_{12} \to \Delta_{12}$, by which $\boldsymbol{M}_1 = \boldsymbol{M}_{\mathrm{red}2}\boldsymbol{M}_{12}\boldsymbol{M}_{\boldsymbol{q},2}$.

6.3 Quantum channels

We have the quantum state space $\mathcal{D}(\mathcal{H})$, (5), the linear maps $\Phi : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H}')$ (completely positive and trace preserving) are called quantum stochastic maps (quantum Markov maps, quantum channels), or bistochastic in the case when it maps white noise to white noise. These can be represented by Kraus operators $K_i : \mathcal{H} \to \mathcal{H}'$ as $\Phi(\rho) =$

$\sum_{i=1}^k K_i\rho K_i^\dagger$ ("completely positive") where $\sum_{i=1}^k K_i^\dagger K_i = I$ ("trace preserving") and if it is additionally bistochastic, then $\sum_{i=1}^k K_i K_i^\dagger = d/d'I$ ("noise preserving").

The deterministic dynamics $\mathcal{D}\to\mathcal{D}$ of a system maps pure states onto pure states.

- Write the Kraus operators $\{K_i\}$ leading to such maps Φ.
- Are they bistochastic?

The dynamics of a closed system is deterministic and also reversible.

- Write the Kraus operators $\{K_i\}$ leading to such maps Φ.
- Are they bistochastic?
- Are reversible maps always deterministic?

A supplementing system (also called "ancilla", described by the state $\omega\in\mathcal{D}(\mathcal{H}_2)$) can be appended to the system in an uncorrelated way, that is, $\Phi_{\sigma,2}:\mathcal{D}(\mathcal{H}_1)\to\mathcal{D}(\mathcal{H}_1\otimes\mathcal{H}_2)$, given as $\Phi_{\sigma,2}(\rho)=\rho\otimes\sigma$.

- Write the Kraus operators $\{K_i\}$ leading to $\Phi_{\sigma,2}$, and also those leading to $\Phi_{\sigma,1}$, given similarly.
- Is it bistochastic?

A subsystem of a composite system can be discarded, that is, $\Phi_{\mathrm{red}2}:\mathcal{D}(\mathcal{H}_1\otimes\mathcal{H}_2)\to\mathcal{D}(\mathcal{H}_1)$, given by the state reduction as $\Phi_{\mathrm{red}2}(\rho_{12})=\mathrm{Red}_2(\rho_{12})$. (So it is the partial trace, $\Phi_{\mathrm{red}2}=\mathrm{Tr}_2$.)

- Write the Kraus operators $\{K_i\}$ leading to $\Phi_{\mathrm{red}2}$, and also those leading to $\Phi_{\mathrm{red}1}$, given similarly.
- Is it bistochastic?

To a system, described by the state ρ, a system of the same kind, described by the state σ can be mixed, that is, $\Phi_{\sigma,x}:\mathcal{D}(\mathcal{H})\to\mathcal{D}(\mathcal{H})$, given as $\Phi_{\sigma,x}(\rho)=x\rho+(1-x)\sigma$.

- Write the Kraus operators $\{K_i\}$ leading to $\Phi_{\sigma,x}$.
- In which case is it bistochastic?

A system can entirely be replaced with a system, described by the state σ (that is, without respect to the original state), that is, $\Phi_\sigma:\mathcal{D}(\mathcal{H})\to\mathcal{D}(\mathcal{H})$, given as $\Phi_\sigma(\rho)=\sigma$ for all ρ.

- Write the Kraus operators $\{K_i\}$ leading to Φ_σ.
- In which case is it bistochastic?

(If you are not fluent enough in using $\otimes$ and the general formalism, you can submit this exercise calculating for qubits, $d=2$. The Kraus form is not unique in general, it is enough to find one solution.)

6.4 Qubit channels

Let us have the state of a qubit given by the density operator $\rho(\boldsymbol{r})$ in the Bloch vector parametrization, see (9). Let us consider the one qubit map

$$\Phi_{\boldsymbol{c}}(\rho)=\frac{1}{2}\sum_{\mu=0}^{3}c_\mu\sigma_\mu\rho\sigma_\mu^\dagger,$$

where $\sigma_0=I$, and the parameters are $\boldsymbol{c}=(c_0,c_1,c_2,c_3)\in\mathbb{R}^4$.

- For which values of the parameters are these channels, that is, trace preserving completely positive (TPCP) maps? (Then these are called *Pauli channels*.)

- For this case, check that these are also bistochastic.
- Write also the canonical Kraus form.

Now we consider some particular cases,
$\boldsymbol{c} = (2, 0, 0, 0)$,
$\boldsymbol{c} = (0, 2, 0, 0)$, $\boldsymbol{c} = (0, 0, 2, 0)$, $\boldsymbol{c} = (0, 0, 0, 2)$,
$\boldsymbol{c} = (2 - x, x, 0, 0)$, $\boldsymbol{c} = (2 - x, 0, x, 0)$, $\boldsymbol{c} = (2 - x, 0, 0, x)$,
$\boldsymbol{c} = 1/2(4 - 3x, x, x, x)$,
$\boldsymbol{c} = 1/2(1 + x, 1 - x, 1 - x, 1 + x)$,
$\boldsymbol{c} = 1/2(1 + x, 1 - x, 1 + x, 1 - x)$,
$\boldsymbol{c} = 1/2(1 + x, 1 - x, 1 - x, 1 - x)$.

- How to imagine these, how do these act on the Bloch ball? That is, for a ρ written in the Bloch vector parametrization (9)

$$\begin{array}{ccc} \varrho & \longmapsto & \rho' = \Phi_{\boldsymbol{c}}(\rho), \\ \boldsymbol{r} & \longmapsto & \boldsymbol{r}' = ? \end{array}$$

- What are the ranges of the parameter x?

Now let us append a constant shift to the usual Pauli channels,

$$\Phi_{\boldsymbol{c},\boldsymbol{d}}(\rho) = \Phi_{\boldsymbol{c}}(\rho) + \frac{1}{2}\boldsymbol{d}\boldsymbol{\sigma},$$

where the additional parameters are $\boldsymbol{d} = (d_1, d_2, d_3) \in \mathbb{R}^3$. It is hard to write in general the range of the parameters $(\boldsymbol{c}, \boldsymbol{d})$, for which the map $\Phi_{\boldsymbol{c},\boldsymbol{d}}$ is a channel.

Now we consider some particular cases,
$\boldsymbol{c} = 1/2((2 - x)^2, 2x - x^2, 0, 0)$, $\boldsymbol{d} = (x, 0, 0)$,
$\boldsymbol{c} = 1/2((2 - x)^2, 0, 2x - x^2, 0)$, $\boldsymbol{d} = (0, x, 0)$,
$\boldsymbol{c} = 1/2((2 - x)^2, 0, 0, 2x - x^2)$, $\boldsymbol{d} = (0, 0, x)$.

- For which values of the parameters x are these channels?
- Then for which values of the parameters x are these bistochastic?
- How to imagine these, how do these act on the Bloch ball?

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7 Measurements

7.1 Indirect classical (generalized) measurements

A measurement device is a set $\{\boldsymbol{M}_i\}$ of substochastic maps, which are positive and sum nonincreasing, such that their sum, $\boldsymbol{M} = \sum_i \boldsymbol{M}_i$ is sum preserving. These can be represented by real $d \times d$ substochastic matrices $\boldsymbol{M}_i$, for which $(\boldsymbol{M}_i)_{jk} \geq 0$ ("positive") and $\sum_{j=1}^d (\boldsymbol{M}_i)_{jk} \leq 1$ ("sum nonincreasing"), and $\sum_{j=1}^d (\boldsymbol{M})_{jk} = \sum_i \sum_{j=1}^d (\boldsymbol{M}_i)_{jk} = 1$ ("sum preserving"). The measurement device is projective, if $\boldsymbol{M}_i = \boldsymbol{P}_i := \boldsymbol{\delta}^i \otimes \boldsymbol{\delta}^{i\dagger}$.

The effect of every generalized measurement can be given as the effect of a projective measurement on an interacting probe system.

• Show this, that is, show that for all $\{\boldsymbol{M}_i\}$ measurement devices on Δ_1 there exist probe system of state $\boldsymbol{q} \in \Delta_{\text{probe}}$, projective measurement device $\{\boldsymbol{P}_i\}$ on Δ_{probe}, and joint reversible dynamics (interaction) $\boldsymbol{R} : \Delta_{1,\text{probe}} \to \Delta_{1,\text{probe}}$, by which $\boldsymbol{M}_i = \boldsymbol{M}_{\text{red probe}}(\boldsymbol{I} \otimes \boldsymbol{P}_i)\boldsymbol{R}\boldsymbol{M}_{\boldsymbol{q},\text{probe}}$.
• Give the indirect representation of a projective measurement, that is for the projective measurement device $\{\boldsymbol{\delta}^i \otimes \boldsymbol{\delta}^{i\dagger}\}$, write $\boldsymbol{q} \in \Delta_{\text{probe}}$, $\{\boldsymbol{P}_i\}$ and $\boldsymbol{R}$ as above, by which $\boldsymbol{\delta}^i \otimes \boldsymbol{\delta}^{i\dagger} = \boldsymbol{M}_{\text{red probe}}(\boldsymbol{I} \otimes \boldsymbol{P}_i)\boldsymbol{R}\boldsymbol{M}_{\boldsymbol{q},\text{probe}}$.

7.2 Qubit generalized measurement

Let us consider a joint quantum system, consisting of two subsystems. The first one is an atom, where we consider only two energy eigenstates of the electronic system. Its Hilbert space is $\mathcal{H}_{\text{Atom}} = \text{Span}\{|\phi_{\text{Atom},0}\rangle \equiv |\text{G}\rangle, |\phi_{\text{Atom},1}\rangle \equiv |\text{E}\rangle\}$, where the two orthonormal vectors stand for the "ground" and "excited" states. The second one is of one photonic mode, which either contains a photon or not. Its Hilbert space is $\mathcal{H}_{\text{Photon}} = \text{Span}\{|\phi_{\text{Photon},0}\rangle \equiv |\text{N}\rangle, |\phi_{\text{Photon},1}\rangle \equiv |\text{P}\rangle\}$, where the two orthonormal vectors stand for the "no photon" and "photon" states. (Here I stress this labelling instead of numerical indexing to emphasize which vector is contained by which Hilbert space in the shorthand notation.)

In a given time interval, the atom in ground state can be excited by the photon with probability p, getting into excited state, absorbing the photon; and in excited state it can emit a photon with the same probability, getting into ground state. (Poisson process.)

We would like to learn the state of the atom. This could be achieved by using the von Neumann measurement given by the orthogonal projectors $\{P_{\text{G}} = |\text{G}\rangle\langle\text{G}|, P_{\text{E}} = |\text{E}\rangle\langle\text{E}|\} \subset \text{Lin}(\mathcal{H}_{\text{Atom}})$, however, this cannot be carried out. The only thing we can measure is wether a photon is emitted or not, that is, the von Neumann measurement given by the orthogonal projectors $\{P_{\text{N}} = |\text{N}\rangle\langle\text{N}|, P_{\text{P}} = |\text{P}\rangle\langle\text{P}|\} \subset \text{Lin}(\mathcal{H}_{\text{Photon}})$ having outcomes "no photon" and "photon" .

Before the measurement, the state of the atom is given by the density operator $\rho_{\text{Atom}} \in \mathcal{D}_{\text{Atom}}$. Let the photonic mode be empty before the measurement, $\rho_{\text{Photon}} = |\text{N}\rangle\langle\text{N}| \in \mathcal{D}_{\text{Photon}}$. Let the effect of the interaction in the time interval be given by the unitary, given by the matrix

$$U : \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-p} & \sqrt{p} & 0 \\ 0 & -\sqrt{p} & \sqrt{1-p} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

written by the natural, product basis $|\text{G}\rangle \otimes |\text{N}\rangle, |\text{G}\rangle \otimes |\text{P}\rangle, |\text{E}\rangle \otimes |\text{N}\rangle, |\text{E}\rangle \otimes |\text{P}\rangle$.
• What is the meaning of the matrix elements? (Why?)
• Is it unitary?

The event "the photodetector fires during the time interval of the interaction" is represented by the projector $P_{\text{P}} = |\text{P}\rangle\langle\text{P}| \in \text{Lin}(\mathcal{H}_{\text{Photon}})$, while the event "the photodetector does not fire during the time interval of the interaction" is represented by the projector $P_{\mathrm{N}} = |\mathrm{N}\rangle\langle\mathrm{N}| \in \mathrm{Lin}(\mathcal{H}_{\mathrm{Photon}})$. Let us write the state of the atom after the time interval of the interaction. This measurement can be given by the following completely positive map, constructed for the outcomes "photon" and "no photon"

$$\Phi_{\mathrm{N}}(\rho_{\mathrm{Atom}}) = \mathrm{Tr}_{\mathrm{Photon}}\Big((I \otimes P_{\mathrm{N}})U(\rho_{\mathrm{Atom}} \otimes \rho_{\mathrm{Photon}})U^\dagger(I \otimes P_{\mathrm{N}})^\dagger\Big),$$
$$\Phi_{\mathrm{P}}(\rho_{\mathrm{Atom}}) = \mathrm{Tr}_{\mathrm{Photon}}\Big((I \otimes P_{\mathrm{P}})U(\rho_{\mathrm{Atom}} \otimes \rho_{\mathrm{Photon}})U^\dagger(I \otimes P_{\mathrm{P}})^\dagger\Big).$$

• Write the matrices of the operators $\Phi_{\mathrm{N}}(\rho_{\mathrm{Atom}}), \Phi_{\mathrm{P}}(\rho_{\mathrm{Atom}}) \in \mathrm{Lin}(\mathcal{H}_{\mathrm{Atom}})$ using the standard basis above.
• Are the maps Φ_{N}, Φ_{P} trace nonincreasing?
Using these, the post measurement states of the atom for the outcomes are

$$\rho_{\mathrm{Atom}} \quad\longmapsto\quad \rho'_{\mathrm{Atom,N}} = \frac{1}{\mathbb{P}(\mathrm{N}|\{\Phi_i\})}\Phi_{\mathrm{N}}(\rho_{\mathrm{Atom}}), \qquad \mathbb{P}(\mathrm{N}|\{\Phi_i\}) = \mathrm{Tr}\big(\Phi_{\mathrm{N}}(\rho_{\mathrm{Atom}})\big),$$
$$\rho_{\mathrm{Atom}} \quad\longmapsto\quad \rho'_{\mathrm{Atom,P}} = \frac{1}{\mathbb{P}(\mathrm{P}|\{\Phi_i\})}\Phi_{\mathrm{P}}(\rho_{\mathrm{Atom}}), \qquad \mathbb{P}(\mathrm{P}|\{\Phi_i\}) = \mathrm{Tr}\big(\Phi_{\mathrm{P}}(\rho_{\mathrm{Atom}})\big).$$

• Write the matrices of these too.
• Does $\mathbb{P}(\mathrm{N}|\{\Phi_i\}) + \mathbb{P}(\mathrm{P}|\{\Phi_i\}) = 1$ hold? (If not, then something is miscalculated…)
• What can we infer from the outcomes ("no photon", "photon") to the state of the atom ("ground", "excited")?
• Write also the effect of the nonselective measurement

$$\rho_{\mathrm{Atom}} \quad\longmapsto\quad \rho'_{\mathrm{Atom}} = \mathbb{P}(\mathrm{N}|\{\Phi_i\})\rho'_{\mathrm{Atom,N}} + \mathbb{P}(\mathrm{P}|\{\Phi_i\})\rho'_{\mathrm{Atom,P}}.$$

• Does $\mathrm{Tr}\,\rho'_{\mathrm{Atom}} = 1$ hold? (If not, then something is miscalculated…)

The measuring device $\{\Phi_{\mathrm{N}}, \Phi_{\mathrm{P}}\}$ can be represented by Kraus operators.
• Find (one particular set of) such Kraus operators, that is, write the matrices of the operators $K_{\mathrm{N}}, K_{\mathrm{P}} \in \mathrm{Lin}\,\mathcal{H}_{\mathrm{Atom}}$ given in

$$\Phi_{\mathrm{N}}(\rho_{\mathrm{Atom}}) = K_{\mathrm{N}}\rho_{\mathrm{Atom}}K_{\mathrm{N}}^\dagger,$$
$$\Phi_{\mathrm{P}}(\rho_{\mathrm{Atom}}) = K_{\mathrm{P}}\rho_{\mathrm{Atom}}K_{\mathrm{P}}^\dagger.$$

(In this particular example it is enough to have only one Kraus operator for each outcome, that is, there is no need for summing up the effects of many Kraus operators for each outcome, as was done in the general case, presented in the lecture, $\sum_j K_{\mathrm{N},j}\rho_{\mathrm{Atom}}K_{\mathrm{N},j}^\dagger$. If you haven't been tired enough yet, you may think over that the reason for this is that the events "no photon" and "photon" are represented by rank one projectors on $\mathcal{H}_{\mathrm{Photon}}$.)
• Does $K_{\mathrm{N}}^\dagger K_{\mathrm{N}} \leq I$ and $K_{\mathrm{P}}^\dagger K_{\mathrm{P}} \leq I$ hold? (If not, then something is miscalculated…) (For self-adjoint operators, $A \leq B$ is defined as $0 \leq B - A$.)

Due to the cyclic property of the trace, the output probabilities are

$$\mathbb{P}(\mathrm{N}|\{\Phi_i\}) = \mathrm{Tr}\big(\Phi_{\mathrm{N}}(\rho_{\mathrm{Atom}})\big) = \mathrm{Tr}\big(K_{\mathrm{N}}^\dagger K_{\mathrm{N}}\rho_{\mathrm{Atom}}\big), \qquad E_{\mathrm{N}} := K_{\mathrm{N}}^\dagger K_{\mathrm{N}},$$
$$\mathbb{P}(\mathrm{P}|\{\Phi_i\}) = \mathrm{Tr}\big(\Phi_{\mathrm{P}}(\rho_{\mathrm{Atom}})\big) = \mathrm{Tr}\big(K_{\mathrm{P}}^\dagger K_{\mathrm{P}}\rho_{\mathrm{Atom}}\big), \qquad E_{\mathrm{P}} := K_{\mathrm{P}}^\dagger K_{\mathrm{P}},$$

that is, those can be obtained by the use of the positive operator valued measure (POVM) $E = \{E_\mathrm{N}, E_\mathrm{P}\} \subset \mathrm{Lin}\,\mathcal{H}_\mathrm{Atom}$.
• Write the matrices of these operators.
• Does $E_\mathrm{N} + E_\mathrm{V} = I_\mathrm{Atom}$ hold? (If not, then something is miscalculated. . .)
• Write also the spectral decomposition of these operators.
These are acting on the Hilbert space $\mathcal{H}_\mathrm{Atom}$, describing the atom, and they are mixtures of the projectors describing the events referring to the atom.
• Think over the meaning of this, that is, how the different possibilities/states of the atom contribute to the measurement of the photon number.

7.3 Induced collapse

Let us have a two-qubit system, with the Hilbert space $\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$, $\dim \mathcal{H}_1 = \dim \mathcal{H}_2 = 2$. We have already seen that a two-qubit state can then be written as

$$\rho_{12} = \frac{1}{4}\Big[I \otimes I + \boldsymbol{r\sigma} \otimes I + I \otimes \boldsymbol{s\sigma} + \boldsymbol{t\sigma} \otimes \boldsymbol{\sigma}\Big],$$

where $\boldsymbol{r}, \boldsymbol{s} \in \mathbb{R}^3$, $\boldsymbol{t} \in \mathbb{R}^3 \otimes \mathbb{R}^3$, and we use the shorthand notation $\boldsymbol{r\sigma} = \sum_{i=1}^3 r^i \sigma_i$, $\boldsymbol{t\sigma} \otimes \boldsymbol{\sigma} = \sum_{ij=1}^3 t^{ij} \sigma_i \otimes \sigma_j$.

As we have seen in the lecture, performing a measurement given by the projectors $P_{1,\pm} = \frac{1}{2}(I \pm \hat{\boldsymbol{v}}\boldsymbol{\sigma})$ (spin measurement along direction $\hat{\boldsymbol{v}} \in S^2 \subset \mathbb{R}^3$, $\|\hat{\boldsymbol{v}}\|^2 = 1$) on subsystem 1 leads to the conditional state of subsystem 2,

$$\rho_2 = \frac{1}{2}(I + \boldsymbol{s\sigma}) \qquad \longmapsto \qquad \rho'_{2,\pm|\hat{\boldsymbol{v}}} = \frac{1}{2}(I + \boldsymbol{s}'\boldsymbol{\sigma}),$$
$$\boldsymbol{s} \qquad \longmapsto \qquad \boldsymbol{s}' = \frac{\boldsymbol{s} \pm \hat{\boldsymbol{v}}\boldsymbol{t}}{1 \pm \hat{\boldsymbol{v}}\boldsymbol{r}},$$

where $(\hat{\boldsymbol{v}}\boldsymbol{t})_j = \sum_{i=1}^3 v_i t_{ij}$ is understood.

• Let us consider the pure state $\rho_{12} = |\psi(\eta)\rangle\langle\psi(\eta)|$, where $|\psi(\eta)\rangle = \sqrt{1-\eta}|01\rangle + \sqrt{\eta}|10\rangle$, where $0 \leq \eta \leq 1/2$, and the computation basis $\sigma_3|j\rangle = (-1)^j|j\rangle$ is used. (We will see later that the "strength" of entanglement in this state is a continuous, monotonically increasing function of η.) Calculate the coefficiens $\boldsymbol{r}, \boldsymbol{s}, \boldsymbol{t}$, and write the Bloch-vector $\boldsymbol{s}'$ of the conditional state above.

• In which cases is $\rho'_{2,\pm|\hat{\boldsymbol{v}}}$ pure? In which cases is $\rho'_{2|\hat{\boldsymbol{v}},\pm}$ maximally mixed? (Can you give a quick proof from general principles?) In the light of this, although it is a nonlinear function of η, what can we say about the Bloch-vector $\boldsymbol{s}'$ of the conditional state?

• How does $\boldsymbol{s}'$ change if we mix some white noise to the system, that is, if we have the initial state $\rho_{12} = (1-w)|\psi(\eta)\rangle\langle\psi(\eta)| + w\frac{1}{2}I \otimes \frac{1}{2}I$ for $0 \leq w \leq 1$?

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8 Bell-nonlocality

8.1 On the CHSH inequality

Let us have the joint system of two qubits, with the Hilbert space $\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$ with $\dim \mathcal{H}_1 = \dim \mathcal{H}_2 = 2$, describing the spin degrees of freedom of two spin-1/2 particles. The operators describing spin measurement of the three orthogonal directions are $S_i = \frac{\hbar}{2}\sigma_i$ (or $\boldsymbol{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$), and the operator describing the spin measurement in direction $\hat{\boldsymbol{v}} \in S^2 \subset \mathbb{R}^3$ ($\|\hat{\boldsymbol{v}}\|^2 = 1$) is $S_{\hat{\boldsymbol{v}}} = \sum_i v_i S_i$ (or $S_{\hat{\boldsymbol{v}}} = \hat{\boldsymbol{v}}\boldsymbol{S}$). The operators of spin measurements of subsystems of the joint system are $S_{\hat{\boldsymbol{v}}} \otimes I$ and $I \otimes S_{\hat{\boldsymbol{u}}}$, while the operator of the spin correlation measurement is $(S_{\hat{\boldsymbol{v}}} \otimes I)(I \otimes S_{\hat{\boldsymbol{u}}}) = S_{\hat{\boldsymbol{v}}} \otimes S_{\hat{\boldsymbol{u}}}$.

- Give the expectation value

$$\langle S_{\hat{\boldsymbol{v}}} \otimes S_{\hat{\boldsymbol{u}}} \rangle = \mathrm{Tr}(\rho_{12} S_{\hat{\boldsymbol{v}}} \otimes S_{\hat{\boldsymbol{u}}})$$

in terms of the measurement directions $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{u}}$ for the pure state $\rho_{12} = |\psi_{12}\rangle\langle\psi_{12}|$, where $|\psi_{12}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ is the singlet state. What if we take the Bell states $|\mathrm{B}_0\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)$, $|\mathrm{B}_1\rangle = \frac{1}{\sqrt{2}}\big(|01\rangle + |10\rangle\big)$, $|\mathrm{B}_2\rangle = \frac{-i}{\sqrt{2}}\big(|01\rangle - |10\rangle\big)$ and $|\mathrm{B}_3\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle - |11\rangle\big)$? (You don't have to write too much for this, if you solved and understood the Bell state exercise earlier.)

- The CHSH inequality (a variant of the Bell inequality), written for the quantum statistics, is

$$\left(\frac{\hbar}{2}\right)^{-2} \big(\langle S_{\hat{\boldsymbol{v}}} \otimes S_{\hat{\boldsymbol{u}}} \rangle + \langle S_{\hat{\boldsymbol{v}}} \otimes S_{\hat{\boldsymbol{u}}'} \rangle + \langle S_{\hat{\boldsymbol{v}}'} \otimes S_{\hat{\boldsymbol{u}}} \rangle - \langle S_{\hat{\boldsymbol{v}}'} \otimes S_{\hat{\boldsymbol{u}}'} \rangle\big) \leq 2,$$

which holds if the expectation values (quantum statistics) can arise from a local hidden variable modell. Find at least one set of measurement direction $\hat{\boldsymbol{v}}, \hat{\boldsymbol{v}}', \hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}' \in S^2$, for which the above inequality is violated.

- (for extra point) Show that the maximal achievable value of the left-hand side is $2\sqrt{2}$.

8.2 More on the CHSH inequality

The content of the previous exercise is pretty much well-known. Now we consider the same setting, but let us weaken the assumptions a bit: can the CHSH inequality be violated by (pure) states not maximally entangled?

- Calculate the expectation value $\langle S_{\hat{\boldsymbol{v}}} \otimes S_{\hat{\boldsymbol{u}}} \rangle$ for the state $|\psi'(\eta)\rangle = \sqrt{1-\eta}|01\rangle - \sqrt{\eta}|10\rangle$, where $0 \leq \eta \leq 1/2$. (We will see later that the "strength" of entanglement is a continuous, monotonically increasing function of η.) What if we take the corresponding η-variants of the Bell states given above?

- For the measurement settings $\hat{\boldsymbol{v}}, \hat{\boldsymbol{v}}', \hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}' \in S^2$, obtained in the previous exercise, find the possible values of η, for which the CHSH inequality is violated. (That is, find the most entangled state in that one-parameter family, for which there can be given local hidden variable description of the scenario, for that particular measurement settings.)

- (for many extra points) Show that for arbitrarily small (but nozero) η, there can be found measurement settings for which the CHSH inequality is violated.