J. Phys. B: At. Mol. Phys. **15** (1982) L179–L184. Printed in Great Britain

LETTER TO THE EDITOR

On the quantum mechanical states of a free charged particle in a uniform magnetic field and a laser field†

J Bergou and S Varró
Central Research Institute for Physics, H-1525 Budapest, PO Box 49, Hungary

Received 4 January 1982

Abstract. The exact solution of the Schrödinger equation for a non-relativistic charged particle simultaneously embedded in both a uniform magnetic field and in a laser field (the latter in the dipole approximation) is obtained in a purely algebraic manner. The relative directions of propagation of the two fields as well as the polarisation state of the laser are arbitrary. The connection with previously known solutions for specific cases is established.

In recent years, considerable interest has been devoted to the theoretical study of charged-particle scattering in the presence of strong radiation fields. This interest has been motivated by the rapid development of high-power lasers, and experiments of this sort are currently in the focus of experimental activity in this field (Ehlotzky 1981). For the description of the phenomena which occur there are essentially two approaches which are accepted. In the case of a laser intensity which is not too high, the usual perturbation theory applies as a means of taking into account the interaction with the laser field. The initial and final states of the scattering are then free-particle states. The scattering by a background potential or by some other mechanism can be treated with the usual tools of scattering theory. In most of the cases this approach leads to a double perturbation theory. For high laser intensities another approach became more popular. The initial and final states are those of a particle embedded in a laser field (exact quantum mechanical states of a charged particle interacting with a radiation field) and the scattering mechanism is treated as a perturbation. This approach clearly requires the exact solution of the free-particle–laser interaction problem (Bergou 1980). Another field of current research interest of a similar nature is charged-particle scattering in the presence of a quantising magnetic field (Ferrante *et al* 1980). This interest is motivated by magnetic confinement experiments in plasmas. It is also possible that this scattering mechanism may find astrophysical applications as well. Motivated by these reasons, of rather different origins, Ferrante *et al* (1979, 1981) recently investigated potential scattering assisted by magnetic and laser fields, since this may play a significant role in both the above mentioned areas. They calculated transitions due to a background potential, $V(r)$, between the exact quantum mechanical states of a particle embedded in both a uniform magnetic field and a laser field. Thus, their method fits in the general scheme of the second of the

† This work was started in Budapest and completed during a short visit of one of the authors (JB) to the Institute for Theoretical Physics, University of Innsbruck, A-6020 Austria. The partial financial support of the Österreichischen Fonds zur Förderung der Wissenschaftlichen Forschung under Contract No 3852 is gratefully acknowledged.

0022-3700/82/060179+06$02.00 © 1982 The Institute of Physics

above mentioned approaches. The treatment, however, was restricted to a particular geometry of the two fields, namely, the polarisation of the laser was directed along the direction of the magnetic field. In this case the solution of the Schrödinger equation factorises. The first factor corresponds to confined planar motion in the plane perpendicular to the direction of the magnetic field and can be described by the states of a two-dimensional harmonic oscillator (see e.g. Sokolov and Ternov 1968). The second factor corresponds to the motion of a free particle moving along the direction of polarisation of the laser light (the z direction) and interacting with the laser field. This factor can be described by a non-relativistic Volkov state (Bergou 1980). The total wavefunction is then simply the product of these two parts.

It is clear that if the polarisation is not strictly directed along the magnetic field, then the motion due to the laser field will be perturbed by the presence of the magnetic field and vice versa. The factorisation property of the wavefunction will then be partially lost and we are left with an intrinsically time-dependent problem. On the other hand, it is also clear that for many physical applications it is precisely this more general field configuration which is the more interesting one. This is the case in actual scattering experiments or in recent free-electron laser arrangements with an axial guiding magnetic field (Freund *et al* 1981). Furthermore, the exact solution of a two-field interaction problem is an interesting quantum mechanical example in its own right. In addition, it might also serve as a starting point for studying the onset of resonance phenomena in more sophisticated physical systems. In view of this, we felt it necessary and useful from the standpoint of immediate applications to present the derivation of the solution in some detail.

For our purposes, we adopt the following model. A particle of charge e and mass m is allowed to interact simultaneously with both a magnetic field and a laser field. The fields are accounted for by the external-field approximation, that is, we describe them by classical electromagnetic potentials. The magnetic field $\boldsymbol{B}=(0,0,B)$ is homogeneous and constant and is directed along the z axis. The laser field is homogeneous but time dependent (dipole approximation). The direction of propagation and the polarisation state of the laser field are arbitrary; thus, in the most general case, it may have components along all three coordinate axes.

The Schrödinger equation for the charged particle embedded in the magnetic and laser fields can be obtained from the field-free Hamiltonian by the usual minimal substitution

$$\hat{p} \to \hat{p} - e\boldsymbol{A}/c$$

and reads

$$\mathrm{i}\hbar \frac{\partial \psi'}{\partial t} = H\psi'$$

where

$$H = \frac{1}{2m}\left(\hat{p} - \frac{e}{c}\boldsymbol{A}(\boldsymbol{r}, t)\right)^2 \qquad \hat{p} \equiv \frac{\hbar}{\mathrm{i}}\nabla \tag{1}$$

with $\boldsymbol{A}(\boldsymbol{r}, t) = \boldsymbol{A}_{\mathrm{L}}(t) + \boldsymbol{A}_B(\boldsymbol{r})$. Here $\boldsymbol{A}_{\mathrm{L}}(t)$ is the vector potential of the laser field and can be written as

$$\boldsymbol{A}_{\mathrm{L}}(t) = (A_x(t), A_y(t), A_z(t)) \tag{1a}$$

and $\boldsymbol{A}_B(\boldsymbol{r})$ is the vector potential of the magnetic field

$$\boldsymbol{A}_B(\boldsymbol{r}) = (-\tfrac{1}{2}By, \tfrac{1}{2}Bx, 0). \tag{1b}$$

Both expressions (1a) and (1b) satisfy the Coulomb gauge condition: div $\boldsymbol{A}_L$ = div $\boldsymbol{A}_B$ = 0. Inserting (1a) and (1b) into (1) it can be seen immediately that the motion along the z axis is still unaffected by the magnetic field. Furthermore, the terms containing $\boldsymbol{A}_L^2$ cause only a uniform overall shift of the Hamiltonian, H, in (1). In view of this one can easily take these two effects (i.e. motion in the z direction and the $\boldsymbol{A}_L^2$ term) into account by the following substitution

$$\begin{aligned}\psi' &= \psi_p(z,t)\psi(x,y,t)\\ \psi_p(z,t) &= \exp\left\{\frac{\mathrm{i}}{\hbar}pz - \frac{\mathrm{i}}{2m\hbar}\int_0^t \mathrm{d}\tau\left[\left(p-\frac{e}{c}A_z(\tau)\right)^2 + \frac{e^2}{c^2}(A_x(\tau)+A_y^2(\tau))\right]\right\}.\end{aligned} \tag{2}$$

The structure of the Schrödinger equation satisfied by ψ is most easily revealed if we introduce the creation and annihilation operators a, a^+ and b, b^+ instead of x, $\hat{p}_x$ and y, $\hat{p}_y$ by the usual definition for a harmonic oscillator (see e.g. Schiff 1968) of frequency $\frac{1}{2}\omega_c$, where $\omega_c \equiv eB/mc$ is the cyclotron frequency. Then for ψ we find the equation

$$\begin{aligned}\mathrm{i}\frac{\partial\psi}{\partial t} = [&\tfrac{1}{2}\omega_c(a^+a+\tfrac{1}{2}) + \tfrac{1}{2}\omega_c(b^+b+\tfrac{1}{2}) + (a+\mathrm{i}b)\alpha/\sqrt{2}\\ &+(a^+-\mathrm{i}b^+)\alpha^*/\sqrt{2} - \mathrm{i}\tfrac{1}{2}\omega_c(ab^+-a^+b)]\psi.\end{aligned} \tag{3}$$

In this expression,

$$\alpha \equiv \frac{e}{c}\frac{\omega_c}{(2m\hbar\omega_c)^{1/2}}(A_y + \mathrm{i}A_x) \tag{3a}$$

and the asterisk denotes complex conjugation. Equation (3) is the Schrödinger equation of two harmonic oscillators (first two terms on the RHS) which are displaced (next two terms) and which are resonantly coupled to each other (last term). Furthermore, the displacement depends on time and is equal in magnitude for the two oscillators 'a' and 'b' but differs in phase by $\frac{1}{2}\pi$. As we will show, both the coupling and displacement terms can be eliminated by appropriately chosen unitary transformations. To demonstrate this, let us first introduce the unitary operator C_θ by the definition

$$C_\theta \equiv \exp[\mathrm{i}\theta(a^+b + ab^+)] \qquad C_\theta^+ = C_\theta^{-1} = C_{-\theta}. \tag{4}$$

This operator has the following properties

$$\begin{aligned}C_\theta^{-1}aC_\theta &= a\cos\theta + \mathrm{i}b\sin\theta\\ C_\theta^{-1}a^+C_\theta &= a^+\cos\theta - \mathrm{i}b^+\sin\theta.\end{aligned} \tag{4a}$$

To obtain the effect of C_θ on b and b^+ one has to exchange a and b and a^+ and b^+ on both sides of (4a). C_θ is thus a rotation in the complex $\{a, b\}$ plane by an angle θ. Let us introduce the transformed wavefunction φ as

$$\varphi = C_\theta^{-1}\psi. \tag{5}$$

If we choose $\theta = -\frac{1}{4}\pi$ we find from equation (3), after elementary algebra

$$\mathrm{i}\frac{\partial\varphi}{\partial t} = [\omega_c(a^+a+\tfrac{1}{2}) + a\alpha + a^+\alpha^*]\varphi. \tag{6}$$

This is the Schrödinger equation of a single displaced harmonic oscillator with a frequency which is twice the frequency of the original individual oscillators. From here we immediately see that the problem is clearly degenerate with respect to the other oscillator since it does not explicitly appear in equation (6). Consequently, $\varphi = \varphi_a \varphi_b$, where φ_b is an arbitrary state of the 'b' oscillators and φ_a still satisfies equation (6). We also mention in this context that the choice $\theta = \frac{1}{4}\pi$ would lead to a similar result for the 'b' oscillators.

To eliminate the interaction terms $(a\alpha + a^+\alpha^*)$ from equation (6), let us introduce another unitary operator (Glauber 1963a, b)

$$D_\sigma \equiv \exp(\sigma a^+ - \sigma^* a) \qquad D_\sigma^+ = D_\sigma^{-1} = D_{-\sigma} \tag{7}$$

which has the displacement property

$$D_\sigma^{-1} a D_\sigma = a + \sigma \qquad D_\sigma^{-1} a^+ D_\sigma = a^+ + \sigma^* \tag{7a}$$

and define a new transformed wavefunction

$$\chi = D_\sigma^{-1} \varphi \exp\left(\mathrm{i} \int_0^t f(\tau)\, \mathrm{d}\tau\right). \tag{8}$$

If we choose

$$\dot{\sigma} + \mathrm{i}\sigma\omega_\mathrm{c} = -\mathrm{i}\alpha^* \qquad f(t) = \mathrm{Re}\,(\sigma(t)\alpha(t)) \tag{8a}$$

then χ satisfies the following Schrödinger equation

$$\mathrm{i}\frac{\partial \chi}{\partial t} = \omega_\mathrm{c}(a^+ a + \tfrac{1}{2})\chi. \tag{9}$$

In the calculation leading from equation (6) to equation (9) we made use of the relation

$$\frac{\partial e^B}{\partial t} = e^B\left(\frac{\partial B}{\partial t} - \frac{1}{2}\left[B, \frac{\partial B}{\partial t}\right]\right)$$

if both B and $\partial B/\partial t$ commute with $[B, \partial B/\partial t]$ (see e.g. Bergou and Varró 1981 where, in addition, much of the above applied disentangling technique is explained in great detail).

The stationary solution of equation (9) is well known

$$\chi_{n_a} = \exp[-\mathrm{i}\omega_\mathrm{c}(n_a + \tfrac{1}{2})t]|n_a\rangle \tag{10}$$

where $|n_a\rangle$ is a number state of the 'a' harmonic oscillator. It is also obvious that any linear combination of the χ_{n_a} states is a solution of (9) and the χ_{n_a} form a complete orthogonal system.

The value of σ for some special cases is as follows:

(a) linear polarisation along the x axis,

$$\boldsymbol{A}_\mathrm{L} = \left(\frac{cE}{\omega}\sin\omega t, 0, 0\right)$$

where E is the amplitude of the laser field strength and ω is the frequency of the laser:

$$\sigma_\mathrm{L} \equiv \tfrac{1}{2}\sigma^+ \exp(\mathrm{i}\omega t) + \tfrac{1}{2}\sigma^- \exp(-\mathrm{i}\omega t)$$

$$\sigma^\pm \equiv -\frac{eE\omega_\mathrm{c}}{\omega(\omega \pm \omega_\mathrm{c})(2m\hbar\omega_\mathrm{c})^{1/2}}$$

(*b*) right-circular polarisation in the xy plane,

$$\boldsymbol{A}_{\mathrm{L}}=\left(-\frac{cE}{\omega}\sin\omega t, \frac{cE}{\omega}\cos\omega t, 0\right) \qquad \sigma_{\mathrm{r}}=\sigma^{+}\exp(\mathrm{i}\omega t)$$

(*c*) left-circular polarisation in the xy plane,

$$\boldsymbol{A}_{\mathrm{L}}=\left(-\frac{cE}{\omega}\sin\omega t, -\frac{cE}{\omega}\cos\omega t, 0\right) \qquad \sigma_{\mathrm{l}}=\sigma^{-}\exp(-\mathrm{i}\omega t).$$

Summarising the above steps, the solution of equation (1) can be written as

$$\psi'=\psi_p(z,t)C_\theta D_\sigma|n_a\rangle|\varphi_b\rangle\exp\left(-\mathrm{i}\int_0^t[\omega_c(n_a+\tfrac{1}{2})+\mathrm{Re}\,(\alpha\sigma)]\,\mathrm{d}\tau\right). \qquad (11)$$

The initial-value problem, associated with equation (1), can be handled in the following manner. First, we note that the non-relativistic Volkov states $\psi_p(z,t)$ form a complete orthogonal system. The method of constructing wave packets from them which match the initial condition in the z direction is explained elsewhere (Bergou 1980). With the rest of the problem (initial value in the xy plane) one can perform the same steps that led from equation (3) to equation (9). One is then able to identify φ_b in equation (6) with the transform of the initial value for the 'b' oscillators and one can construct a linear combination of the χ_{n_a} functions to match the initial condition for the 'a' oscillators.

Finally, we mention two more points. For certain applications another form of the solution can be more convenient, namely, if we perform the displacement transformation first on equation (3). In this case the two displacement terms are eliminated and we are left with the well known problem of a particle moving in a constant magnetic field (Sokolov and Ternov 1968). This type of solution, together with its application to charged-particle scattering, is left for a separate publication (Bergou *et al* 1981). Another interesting point is that the form of the vector potential given by equation (1*b*) corresponds to a specific gauge. We can always perform a gauge transformation of the type

$$\boldsymbol{A}'_B=\boldsymbol{A}_B+\mathrm{grad}\,\chi \qquad \nabla^2\chi=0. \qquad (12)$$

The choice

$$\chi=\tfrac{1}{2}Bxy \qquad (12a)$$

makes $\boldsymbol{A}'_B$ independent of y. If we perform a corresponding unitary transformation of the wavefunction in equation (1)

$$\psi'=\exp\left(-\frac{\mathrm{i}}{\hbar}\frac{e}{c}\chi\right)\psi'' \qquad (12b)$$

then ψ'' still satisfies a Schrödinger equation of type (1) but with $\boldsymbol{A}_B(\boldsymbol{r})$ replaced by $\boldsymbol{A}'_B(\boldsymbol{r})$. Since the Hamiltonian is now independent of y the solution can be an eigenstate of $\hat{p}_y$ with eigenvalue p_y. The eigenvalues of H will now be continuously degenerate with respect to p_y, whereas in the unprimed gauge the degeneracy was discrete. These two types of basis sets are equivalent since they are connected with the unitary transformation (12*b*). It may turn out, however, that for certain applications this last set of solutions is more convenient.

Other aspects of the problem (further discussion of (11), solution for more general potentials, etc) are to be published separately (Bergou and Ehlotzky 1982).

References

Bergou J 1980 *J. Phys. A: Math. Gen.* **13** 2817
Bergou J and Ehlotzky F 1982 *J. Phys. B: At. Mol. Phys.* **15** L185
Bergou J, Ehlotzky F and Varró S 1981 *Phys. Rev.* A submitted
Bergou J and Verró S 1981 *J. Phys. A: Math. Gen.* **14** 1469
Ehlotzky F 1981 *Can. J. Phys.* **59** 1200
Ferrante G, Nuzzo S and Zarcone M 1979 *J. Phys. B: At. Mol. Phys.* **12** L437
—— 1981 *J. Phys. B: At. Mol. Phys.* **14** 1565
Ferrante G, Nuzzo S, Zarcone M and Bivona S 1980 *J. Phys. B: At. Mol. Phys.* **13** 731
Freund H P, Sprangle P, Dillenburg D, da Jornada E H, Liberman B and Schneider R S 1981 *Phys. Rev.* A **24** 1965
Glauber R J 1963a *Phys. Rev.* **130** 2529
—— 1963b *Phys. Rev.* **131** 2766
Schiff L I 1968 *Quantum Mechanics* (New York: McGraw-Hill)
Sokolov A A and Ternov I M 1968 *Synchrotron Radiation* (New York: Pergamon)