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Entangled photon–electron states and the number-phase minimum uncertainty states of the photon field

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Abstract. The exact analytic solutions of the energy eigenvalue equation of the system consisting of a free electron and one mode of the quantized radiation field are used for studying the physical meaning of a class of number-phase minimum uncertainty states. The states of the mode which minimize the uncertainty product of the photon number and the Suskind and Glogower (1964 *Physics* **1** 49–61) cosine operator have been obtained by Jackiw (1968 *J. Math. Phys.* **9** 339–46). However, these states have so far remained mere mathematical constructions without any physical significance. It is proved that the most fundamental interaction in quantum electrodynamics—namely the interaction of a free electron with a mode of the quantized radiation field—leads quite naturally to the generation of the mentioned minimum uncertainty states. It is shown that from the entangled photon–electron states developing from a highly excited number state, due to the interaction with a Gaussian electronic wave packet, the minimum-uncertainty states of Jackiw's type can be constructed. In the electron's coordinate representation, the physical meaning of the expansion coefficients of these states is the joint probability amplitudes of simultaneous detection of an electron and of a definite number of photons. The photon occupation probabilities in these states preserve their functional form as time elapses, but they depend on the location in space-time of the detected electron. An analysis of the entanglement entropies derived from the photon number distribution and from the electron's density operator is given.

Contents

1. Introduction

Entanglement and non-locality in quantum mechanics were first discussed by Einstein *et al* (1935), and their main conclusion was that quantum mechanics is not a 'complete theory', because not all 'elements of physical reality' have a counterpart in the theory. As Bohm (1951) writes in his book at the beginning of section 22.15, 'Their criticism has, in fact shown to be unjustified [see Bohr (1935)], and based on assumptions concerning the nature of matter which implicitly contradict the quantum theory at the outset.' Motivated by the above work of Einstein, Podolsky and Rosen (EPR), Schrödinger (1935a, 1935b, 1935c) presented a detailed study of the conceptual aspects of quantum mechanics. In this series of papers, he introduced the famous 'Schrödinger cat' and the concept of 'entanglement' ('Verschränkung' in Schrödinger's terminology). In his book, in section 22.15, Bohm (1951) analyses the 'EPR-paradox' in detail by considering a disintegration of a quantum system (a molecule having initially zero-spin angular momentum) consisting of two spin-$\frac{1}{2}$ atoms, and determines the correlations of the spin directions observed at spatially separated detectors. The first reliable experiments, proposed by Wheeler (1946) in this context, were carried out by Wu and Shaknov (1950), in which they measured coincidence counting rates at different relative azimuths of the polarization of two γ-rays, stemming from electron–positron pair annihilation, and detected by two opposing scintillation counters. They found that the counting rates of perpendicular polarization were two times larger than the rates of parallel polarization. In the optical regime, the first experimental realization of the 'Einstein–Podolsky–Rosen–Bohm Gedankenexperiment' was achieved much later by Aspect *et al* (1982a, 1982b). They measured the linear-polarization correlation of pairs of photons emitted in a radiative cascade of calcium, and found excellent agreement with the quantum mechanical predictions, and the greatest violation of generalized Bell's inequalities at that time. Concerning Bell's inequalities, see e.g. the references in Aspect *et al* (1982a, 1982b) and Wigner (1970) and the references therein. In the meantime, it turned out that entanglement plays a crucial role in the nowadays rapidly developing branches of quantum physics and informatics, namely in quantum information theory (see e.g. Alber *et al* 2001, Bouwmeester *et al* 2001 and Stenholm and Suominen 2005) and in quantum computing and quantum communication (see e.g. Williams 1999 and Nielsen and Chuang 2000).

In the above-mentioned examples, the entangled particles are of the same sort. In the present paper, we shall discuss entanglement between photons and electrons. It will turn out that the entangled photon–electron states, to be constructed in section 4, have a close connection with the 'critical states' introduced by Jackiw (1968), which minimize a number-phase uncertainty product of the photon field. That is why, concerning the problem of the phase operator of a mode of the quantized electromagnetic radiation, we think that, for the sake of completeness of the present paper, it is instructive to give a brief summary of the basic references dealing with this subject.

In his pathbreaking paper on the quantum theory of emission and absorption of radiation, Dirac (1927) introduced the photon absorption and emission operators in the form $b_r = \mathrm{e}^{-\mathrm{i}\theta_r/h} N_r^{1/2}$ and $b_r^* = N_r^{1/2}\,\mathrm{e}^{\mathrm{i}\theta_r/h}$, respectively, where, in his notation, r is the mode index, h is Planck's constant divided by 2π and * denotes Hermitian conjugation. The number (action) operators N_r and the canonically conjugate angle operators θ_r are assumed to satisfy the Heisenberg commutation relation, and, as a consequence, $b_r b_r^* - b_r^* b_r = 1$. We note here that in the present paper, we shall use the following notations for one mode: $b \to A$, $b^* \to A^+$; thus $[A, A^+] \equiv AA^+ - A^+A = 1$. The 'polar decompositions' used by Dirac are replaced by the relations $A = EN^{1/2}$ and $A^+ = N^{1/2}E^+$, as will be discussed in more detail in section 2. In the same year when Dirac's paper appeared, London (1927) published his study on the angle variables and canonical transformations in quantum mechanics. He proved that though the ladder operators E and E^+ have a well-defined matrix representation, they cannot be expressed as an exponential of the form $\mathrm{e}^{\pm\mathrm{i}\Phi}$, where Φ would be a Hermitian matrix. It is sure that Dirac was aware of this discrepancy. According to Jordan (1927), in a conversation with him, Dirac remarked that the possibility to derive many correct results by using the *formal* relation $[N, \Phi] = \mathrm{i}$ comes from the fact that the correct relation $[E, N] = E$ has been implicitly used, in fact, instead of the former one, in all the derivations of the results. Formally, the correct relations $ENE^+ = N + 1$ and $E^+NE = N - 1$ can also be reproduced by assuming $E = \mathrm{e}^{\mathrm{i}\Phi}$ and $E^+ = \mathrm{e}^{-\mathrm{i}\Phi}$ with a Hermitian Φ, satisfying the commutation relation $[N, \Phi] = \mathrm{i}$. At this point, let us note that the above-discussed problem of the quantum-phase variable does not show up in the case of quantization of the canonically conjugate pair angle and orbital momentum of a planar motion (because the spectrum does not terminate at zero angular momentum), as is illustrated in the extensive and thorough study by Kastrup (2006b), which appeared recently.

The non-existence of a Hermitian phase operator of a harmonic oscillator was rediscovered by Susskind and Glogower (1964). They introduced the Hermitian 'cosine' and 'sine' operators, whose basic properties will be briefly summarized in section 2 of the present paper. In their extensive review paper on phase and angle variables in quantum mechanics, Carruthers and Nieto (1968) derived a couple of number-phase uncertainty relations by using the cosine and sine operators, and Jackiw (1968) constructed a 'critical state' which minimizes one of these uncertainty products. Garrison and Wong (1970) constructed a quantum analogon of the classical periodic phase function (saw-tooth), which satisfies the Heisenberg commutation relation with the number operator on a dense set of the Hilbert space of the oscillator. Moreover, they have constructed the eigenstates of this periodic phase operator. In our opinion, this was the first mathematically correct approach towards the solution of the original problem of quantum phase. Paul (1974) has proposed an alternative description of the phase of a microscopic electromagnetic field, and discussed the possibilities of its measurement.

A new impetus was given to the study of the quantum-phase problem after the paper of Pegg and Barnett (1989) appeared. They truncated the state space of the harmonic oscillator, and

were able to construct a Hermitian phase operator on this finite-dimensional Hilbert space. We mention that the possibility of using a finite-dimensional (truncated) Hilbert space in this context has already been discussed by Jordan (1927). The approach of Pegg and Barnett (1989) was refined by Popov and Yarunin (1992). The limit matrix elements of the phase operator in number representation (as we let the dimension of the Hilbert space go to infinity) obtained by these authors have already been presented by Weyl (1931). We note that, seemingly, none of the above authors, publishing their papers since the 1960s, had known about the fundamental early works of London (1926, 1927). The phase distribution of highly squeezed states has been determined by Schleich *et al* (1989) (where the reference to London's work first appeared in the modern era) by using the quantum phase-space distribution (the Wigner function) of the quantized mode (see also the book by Schleich (2001), in particular chapters 8 and 13). The problem of quantum-phase measurements has been discussed by Shapiro and Shepard (1991), partly on the basis of 'normalizable phase states'. The question of operators of phase has been thoroughly analysed by Bergou and Englert (1991) both from the formal point of view and from the physical point of view. In a series of papers, Noh *et al* (1991, 1992a, 1992b, 1993) have studied both theoretically and experimentally the quantum-phase dispersion on the basis of their operationally defined cosine and sine operators. In their scheme, these definitions are based on measured photon number counts in an eight-port interferometer. Freyberger and Schleich (1993) have performed an analysis of a similar phase operator along with the experiment by Noh *et al* (1991) by using radially integrated phase-space distributions. In this context, see also the thoroughly written dissertation by Freyberger (1994), and references therein. In the meantime, an ample literature has been accumulated concerning the quantum-phase problem. For further reading and references, we refer the reader to the topical issue of *Physica Scripta* edited by Schleich and Barnett (1993), in which also some historical aspects are summarized by Nieto (1993). See also the critical review by Lynch (1995) and the book by Peřinová *et al* (1999) on the description of phase in optics. Concerning the recent developments of the concept of quantum phase of a linear oscillator, see the thorough group theoretical studies by Kastrup (2003, 2006a, 2007), in which a genuinely new approach to this problem has been worked out.

In the present paper, it is proved that the most fundamental interaction in quantum electrodynamics (QED)—namely the interaction of a free electron with a mode of the quantized radiation field—leads quite naturally to the generation of the above-mentioned number-phase minimum uncertainty states. We emphasize that here we are merely dealing with non-relativistic quantum mechanics, where the interaction of the electron with the quantized mode is represented by the minimal coupling term between a free charged particle and an oscillator. The analysis to be presented here is restricted to the study of the interaction of one Schrödinger electron with one quantized mode of the radiation field. Neglecting the interaction with other modes is justified by the fact that we assume a very highly occupied single mode. Thus, in fact, we are not using complete field operators used in the very QED. In section 2, we briefly summarize the basic properties of the Susskind and Glogower (1964) 'cosine' and 'sine' operators, and we give the associated number-phase uncertainty relations and present the 'critical state' found by Jackiw (1968), which minimizes one of the uncertainty products. In section 3, we present the exact stationary solutions of the photon–electron system, in which the interaction is taken into account up to infinite order. In section 4, we shall construct the entangled photon–electron states on the basis of these stationary states. It will be shown that the entangled photon–electron states developing from a highly excited number state due to the interaction with a Gaussian electronic wave packet have the same functional form as the 'critical states' derived

by Jackiw (1968). In section 5, we derive the reduced density operators of the photon and of the electron. On the basis of these reduced density operators, various entanglement entropies are calculated. In section 6, a short summary closes our paper. The mathematical details of the derivation of our results are presented in appendices A and B.

2. The number-phase minimum uncertainty states of Jackiw

The number-phase uncertainty product (in contrast to the usual Heisenberg uncertainty products, which are valid e.g. for the variances of the *Cartesian* components of the momentum and position of a particle)

$$(\Delta N)^2(\Delta\Phi)^2 \geqslant 1/4 \tag{1}$$

cannot have a well-defined mathematical meaning for a generic state of a quantized mode of the electromagnetic radiation. This is because Φ itself cannot be represented by a matrix (or an operator), as London (1927) has already shown long ago. Equation (1) would be valid if a Heisenberg commutation relation $[N, \Phi] = \mathrm{i}$ existed for the number operator N and for the phase operator Φ, which is not the case here. That is the reason why Carruthers and Nieto (1968) proposed other uncertainty products given in terms of the C ('cosine') and S ('sine') operators introduced by Susskind and Glogower (1964),

$$C \equiv (E + E^+)/2, \qquad S \equiv (E - E^+)/2\mathrm{i}, \tag{2}$$

which are well-defined operators. Here, E is the so-called 'exponential phase operator' defined by the '*polar decomposition* of the photon absorption operator A' (which would be the quantum analogon of the polar decomposition of a complex number, $z = \mathrm{e}^{\mathrm{i}\varphi}\sqrt{z^*z}$):

$$\begin{gathered} A = E\sqrt{N}, \qquad N = A^+A, \qquad E = \sum_{k=0}^{\infty} |k\rangle\langle k+1|, \\ A^+ = \sqrt{N}E^+, \qquad E^+ = \sum_{k=0}^{\infty} |k+1\rangle\langle k|. \end{gathered} \tag{3}$$

We note that the 'exponential phase operator', the ladder operator E, was originally introduced by London (1926) and used by Jordan (1927), too. In equation (3), $\{|k\rangle,\ k = 0, 1, 2, \ldots\}$ is the complete orthonormal set of eigenstates of the photon number operator (i.e. $N|k\rangle = k|k\rangle$), serving as a countable basis set of the Hilbert space of the mode under discussion (i.e. $\sum_{k=0}^{\infty} |k\rangle\langle k| = 1$ is the identity operator). Then, owing to the equations $[E, N] = E$ and $[E^+, N] = -E^+$, the following commutation relations can be derived for N, C and S:

$$[N, C] = -\mathrm{i}S, \qquad [N, S] = \mathrm{i}C, \qquad [S, C] = P_0/2\mathrm{i}, \tag{4}$$

where $P_0 \equiv |0\rangle\langle 0|$ is the projector of the vacuum state of the mode, for which $A|0\rangle = 0$. As we see, the 'cosine' and the 'sine' operators C and S, respectively, do not commute, because they cannot be expressed in terms of exponentials of a common (Hermitian) operator Φ in the form $\mathrm{e}^{\pm\mathrm{i}\Phi}$. The reason for this is that the 'exponential phase operator' E, introduced in equation (3), is not unitary but only 'half-unitary' (called 'partially isometric' in mathematical terminology, see e.g. Riesz and Szőkefalvi-Nagy (1965), sections 109 and 110). Really, $EE^+ = 1$ holds, but, on the other hand, $E^+E = 1 - \mathrm{P}_0$, and, moreover, as a consequence of the half-unitary property of E the sum of the squares of the 'cosine' and 'sine' operators is not equal to unity, $C^2 + S^2 = 1 - P_0/2 \neq 1$. We mention that for large coherent excitations of the mode, the moments of

C and S have a similar form as the moments of the ordinary c-number cosine and sine functions. We have to note here that Kastrup (2006a) has recently raised serious objections against the use of the Susskind and Glogower cosine and sine operators in the description of quantal phase properties of the linear oscillator. On the basis of the analysis presented in section 5 of his paper, he concludes that 'the London–Susskind–Glogower operators $\tilde{C}_k$ and $\tilde{S}_k$ are *not* appropriate for measuring angle properties of a state!'. We would like to emphasize that in the present study we are not concerned with the question whether the operators C and S, defined in equation (2), are suitable or not suitable to characterize the quantal phase properties. We merely show that states of essentially the same mathematical structure as that of the 'minimizing states' constructed by Jackiw (1968) may be generated in non-pertubative photon–electron interactions in the strong field regime. Thus, we shall not discuss the (questionable or non-existing) physical relevance of C and S themselves in the context of the problem of quantal phase.

The uncertainty products associated with the above commutation relations, equation (4), are the following (Carruthers and Nieto 1965, 1968):

$$U_1(\Psi) \equiv (\Delta N)^2(\Delta C)^2/\langle S\rangle^2 \geqslant 1/4, \quad U_2(\Psi) \equiv (\Delta N)^2(\Delta S)^2/\langle C\rangle^2 \geqslant 1/4. \tag{5}$$

In the above equations, Ψ refers to the state of the quantized mode of the electromagnetic field under discussion, and $(\Delta N)^2$, $(\Delta C)^2$ and $(\Delta S)^2$ are the variances in that state. Jackiw (1968) has constructed a 'critical state' which minimizes the first of these uncertainty products, $U_1(\Psi)$,

$$|\Psi\rangle = \sum_{n=0}^{\infty} a_n|n\rangle = \kappa \sum_{n=0}^{\infty}(-\mathrm{i})^n I_{n-\nu}(\gamma)|n\rangle, \tag{6}$$

where I_n is a modified Bessel function of the first kind of order n (see the definition in Gradshteyn and Ryzhik (2000), formula 8.406.3), and κ is a normalization factor. The parameter $\nu \equiv \langle N\rangle$ denotes the mean photon number, $\gamma \equiv \langle S\rangle \neq 0$ and the case in which $\langle C\rangle \equiv \langle\Psi|C|\Psi\rangle = 0$ has been considered. The expansion coefficients a_n have been determined from the recursion relations $(\nu - n)a_n = (\mathrm{i}\gamma/2)(a_{n-1} + a_{n+1})$, coming from the minimizing condition, by taking the subsidiary condition $a_{-1} = 0$ into account. The last requirement (which is equivalent to the equation $I_{-1-\nu}(\gamma) = 0$) forces ν to satisfy $2s < \nu < 2s + 1$, where $s = 0, 1, \ldots$. We have found that this requirement is a consequence of the second theorem of Hurwitz on the zeros of Bessel functions (see Watson 1944, section 15.27). The states which allow $U_2(\Psi)$ to reach $1/4$ can also be constructed, by using the same method. Jackiw (1968) has noted on the states given by equation (6) that 'unfortunately these states do not seem to have any physical significance'. In the present paper, we will show that states of the same structure as that of $|\Psi\rangle$ naturally appear in the non-perturbative analysis of the simplest interaction of QED (namely, the interaction of a free electron with a quantized mode of the electromagnetic radiation). Thus, on the basis of our analysis, we may say that the states to be constructed below have a fundamental significance.

3. Exact energy eigenstates of the interacting photon–electron system

In order to make our paper self-contained, in the present section we briefly summarize the basic steps towards the determination of the exact energy eigenstates of the interacting photon–electron system. We mention that the interaction of electrons with a quantized electromagnetic field within a conducting enclosure has been treated by Smith (1946), but

he used perturbation theory, and then rate equations, to treat higher order processes. In his pioneering work on the connection of communication theory and quantum physics, Gabor (1950) also studied a similar system (the transit of electrons in a wave guide), though he used semiclassical pertubation theory and a different geometry.

Let us consider the energy eigenvalue equation of the joint interaction of a quantized mode of the radiation field with a Schrödinger electron. For the sake of simplicity, we take for the mode a circularly polarized plane wave in dipole approximation. In this case, we do not get squeezing in the stationary states, because the interaction coming from the A^2 term of the Hamiltonian is diagonal. The exact solution of the Dirac equation of the system consisting of an electron and a quantized plane wave mode of the radiation field was first presented by Bersons (1969) long ago. In this pioneering work, Bersons (1969) used the 'coordinate representation' $A = 2^{-1/2}(\xi + \partial/\partial\xi)$ and $A^+ = 2^{-1/2}(\xi - \partial/\partial\xi)$ for the photon annihilation and creation operators, as has also been done by Bloch and Nordsieck (1937). Later, without relying on a concrete representation of the photon annihilation and creation operators, the complete discussions for a Schrödinger electron and for a Dirac electron have been published by Bergou and Varró (1981a, 1981b), and have been applied to determine non-perturbatively the cross sections of multiphoton bremsstrahlung and multiphoton Compton scattering. Concerning the question of squeezing in photon–electron systems, see e.g. Bergou and Varró (1981a), Ben-Aryeh and Mann (1985) and Becker *et al* (1987). We will consider here only the (simplest) Schrödinger case, and study the interaction with a circularly polarized mode in dipole approximation. The energy eigenvalue equation now reads

$$\left[\frac{1}{2m}\left(\hat{\vec{p}} + \frac{e}{c}\vec{A}\right)^2 + H_f\right]|\psi_{\vec{p},n_0}\rangle = E_{\vec{p},n_0}|\psi_{\vec{p},n_0}\rangle, \tag{7}$$

where the vector potential is given as

$$\vec{A} = a(\vec{\varepsilon}A + \vec{\varepsilon}^* A^+), \quad \text{where } a \equiv (2\pi\hbar c^2/\omega L^3)^{1/2}, \tag{8}$$

and $\vec{\varepsilon} = (\vec{\varepsilon}_x + \mathrm{i}\vec{\varepsilon}_y)/\sqrt{2}$ is the complex polarization vector (for right circular polarization, when the field is assumed to be perpendicular to the z-direction), ω is the circular frequency of the mode and L^3 is the quantization volume. $H_\mathrm{f} = \hbar\omega(A^+A + 1/2)$ is the bare field energy. $-e$, m and c have their usual meaning: the electron's charge and mass, and the velocity of light in vacuum, respectively. $\hbar$ denotes Planck's constant divided by 2π. In equation (7), $|\psi_{\vec{p},n_0}\rangle$ are exact stationary states of the interacting photon–electron system characterized by two quantum numbers $\vec{p}$ (the electron's momentum) and n_0 (a non-negative integer, which, by switching off the interaction, reduces to the initial photon occupation number). $E_{\vec{p},n_0}$ are the corresponding energy eigenvalues.

The Hamiltonian on the left-hand side of equation (7) can be rewritten as

$$\begin{aligned} H &= \frac{\hat{\vec{p}}^2}{2m} + \hbar\Omega(A^+A + 1/2) + \frac{ea}{mc}\hat{\vec{p}}\cdot(\vec{\varepsilon}A + \vec{\varepsilon}^* A^+), \\ \Omega &\equiv \omega(1 + \omega_\mathrm{p}^2/2\omega^2), \\ \omega_\mathrm{p}^2 &= 4\pi e^2/mL^3. \end{aligned} \tag{9}$$

Notice that ω_p is formally nothing else but the plasma frequency for an electron density $1/L^3$. In obtaining equation (9) we have taken into account that $\vec{\varepsilon}\cdot\vec{\varepsilon} = 0$, $\vec{\varepsilon}^*\cdot\vec{\varepsilon}^* = 0$ and $\vec{\varepsilon}^*\cdot\vec{\varepsilon} = 1$. The linear interaction term on the right-hand side of equation (9) can easily be transformed out from

the eigenvalue equation, equation (7), by applying the following displacement operator with a properly chosen parameter σ:

$$D[\sigma(\vec{p})] = \exp[\sigma^*(\vec{p})A - \sigma(\vec{p})A^+] \quad \text{with } \sigma(\vec{p}) = -(ea/mc\hbar\Omega)\vec{p}\cdot\vec{\varepsilon}^*. \tag{10}$$

We note that the displacement operators of the form displayed in equation (10) have an important role in the quantum theory of optical coherence and coherent states, as was first shown by Glauber (1963a, 1963b) in his pathbreaking papers. Such displacement operations were also used much earlier by Bloch and Nordsieck (1937) in their fundamental study of the problem of infrared divergences in QED, in order to transform out the interaction terms. By applying the displacement operation we receive a transformed Hamiltonian that is diagonal in both the electron and the photon variables; hence its eigensolutions can be written down as simple products of the type $|\vec{p}\rangle|n\rangle$, where $|\vec{p}\rangle$ is a momentum eigenstate of the electron. Accordingly, we obtain the eigensolutions of the original Hamiltonian, equation (9), in the form

$$|\psi_{\vec{p},n_0}\rangle = |\vec{p}\rangle D[\sigma(\vec{p})]|n_0\rangle. \tag{11}$$

Equation (11) shows that the stationary states of the photon–electron system are products of momentum eigenstates of the electron and generalized coherent states of the photon. If $n_0 = 0$, then the solutions have the structure $|\vec{p}\rangle|\sigma\rangle$, where $|\sigma\rangle$ is an ordinary coherent state. Thus, one may say that (at least, according to the present very simplified description) the self-radiation field of the electron is in a coherent state. The complete stationary solutions (being solutions of the time-dependent Schrödinger equation of the joint system) read

$$|\psi_{\vec{p},n_0}(t)\rangle = |\psi_{\vec{p},n_0}\rangle \exp[-\mathrm{i}E_{\vec{p},n_0}t/\hbar], \tag{12}$$

where the energy eigenvalues can be brought to the form

$$E_{\vec{p},n_0} = \frac{p_\perp^2}{2m_\perp} + \frac{p_z^2}{2m} + \hbar\Omega(n_0 + 1/2) \quad \text{with } m_\perp = \frac{1+\omega_\mathrm{p}^2/2\omega^2}{1-\omega_\mathrm{p}^2/2\omega^2}m. \tag{13}$$

In equation (13), we have used the transverse components $(p_x, p_y) = p_\perp(\cos\chi, \sin\chi)$ of the electron's momentum. It is interesting to note that the 'transverse mass' $m_\perp$ given in the second equation of equation (13) can in principle be negative (if $\omega_\mathrm{p}^2/2\omega^2 > 1$); thus the total energy of the system can also be negative in a certain parameter range, which would mean a sort of 'attractive interaction' ('bound states') of the mode and of the electron. On the other hand, according to the definition of the one-electron plasma frequency in equation (9), for a large enough quantization volume L^3, $\omega_p \ll \omega$; thus $m_\perp$ practically equals the original bare mass m. We shall not discuss this question any further in the present paper. For simplicity, in the following we will always assume that $\omega_\mathrm{p}^2/2\omega^2 < 1$; thus, the 'transverse mass' $m_\perp$ is positive. It is clear that if the ratio $\omega_\mathrm{p}^2/2\omega^2$ approaches 1 from below, then $m_\perp$ can be much larger than the bare mass m of the electron. For later convenience, we rewrite equation (12) in the form

$$|\psi_{\vec{p},n_0}(t)\rangle = |p_z\rangle \exp\left[-\frac{\mathrm{i}}{\hbar}\frac{p_z^2}{2m}t - \mathrm{i}(n_0+1/2)\Omega t\right]|\psi_\perp(t)\rangle, \tag{14}$$

$$|\psi_\perp(t)\rangle \equiv |\vec{p}\rangle \exp\left[-\frac{\mathrm{i}}{\hbar}\frac{p^2}{2m_\perp}t\right] D[\sigma(\vec{p})]|n_0\rangle. \tag{15}$$

In order to simplify the notation, in equation (15) the symbol $\vec{p} \equiv (p_x, p_y)$ has been used for the transverse momentum of the electron, i.e. $\vec{p} \equiv (p_x, p_y) \equiv p(\cos\chi, \sin\chi) = p_\perp(\cos\chi, \sin\chi)$. We note that, owing to the unitarity of the displacement operators, equation (10), the exact

solutions given by equation (11) form a complete orthogonal set on the product Hilbert space $H_{\text{photon}} \otimes H_{\text{electron}}$ associated with the interacting photon–electron system. The photon statistics of the generalized coherent state of the type $D[\sigma]|n\rangle$, given on the right-hand side of equation (11), is governed by the matrix elements

$$c_{k,n} \equiv \langle k|D[\sigma]|n\rangle = \begin{cases} (n!/k!)^{1/2}\sigma^{k-n}L_n^{k-n}(|\sigma|^2)\mathrm{e}^{-|\sigma|^2/2}, & (k \geqslant n), \\ (k!/n!)^{1/2}(-\sigma^*)^{n-k}L_k^{n-k}(|\sigma|^2)\mathrm{e}^{-|\sigma|^2/2}, & (0 \leqslant k < n), \end{cases} \tag{16}$$

where L_n^s denote generalized Laguerre polynamials (for the definition of them, see e.g. Gradshteyn and Ryzhik (2000), formula 8.970.1). To our knowledge, the matrix elements of the type given in equation (16) were first published in the work by Bloch and Nordsieck (1937), which we have already quoted before. Later, Schwinger (1953) derived such matrix elements in one of his famous series of papers on the theory of quantized fields, and they also appear in his study on the Brownian motion of a quantum oscillator (Schwinger 1961). For further details, see e.g. Bergou and Varró (1981a, 1981b). The expectation value of the photon number $\langle k\rangle$, and its variance can be calculated either directly from equation (16) or by using the displacement properties $D^+(\sigma)AD(\sigma) = A+\sigma$ and $D^+(\sigma)A^+D(\sigma) = A+\sigma^*$,

$$\begin{aligned} \langle k\rangle &= \sum_{k=0}^{\infty}|c_{k,n_0}|^2 k = \langle n_0|D^+(\sigma)A^+AD(\sigma)|n_0\rangle = n_0+|\sigma|^2, \\ \Delta k^2 &\equiv \langle k^2\rangle - \langle k\rangle^2 = (2n_0+1)|\sigma|^2. \end{aligned} \tag{17}$$

4. Entangled photon–electron states

In the present section, it is proved that the interaction of a free electron with a mode of the quantized radiation field leads to the generation of the number-phase minimum uncertainty states discussed in section 2. It is shown that the entangled photon–electron states developing from a highly excited number state due to the interaction with a Gaussian electronic wave packet have the same functional form as the minimum 'critical states' found by Jackiw (1968). In the electron's coordinate representation, the expansion coefficients of these states are expressed in terms of modified Bessel functions of the first kind (as has been shown in equation (6)) whose argument now depends on the electron's coordinate. The photon statistics of these states preserve their functional form as time evolves, but the occupation probabilities depend on the spatio-temporal position of the electron's detection. We note that on this subject, preliminary results have already long been presented by us (Varró 2000), but we have not published them until now.

According to equations (10), (11) and (14), only the transverse motion of the electron couples with the radiation field; thus the longitudinal motion is merely a free propagation. In the following, we shall not discuss any further this longitudinal dynamics, but, rather, we concentrate on the study of the transverse part of the wave packet dynamics, which represents in our case the interaction of the electron and the quantized mode of the radiation field. The entangled photon–electron states developing from a number state due to the interaction with an electronic wave packet have the form

$$|\psi\rangle = \int \mathrm{d}^2 p g(\vec{p})|\psi_\perp(t)\rangle, \quad \text{with } g(\vec{p}) \equiv g(p) = (w/\hbar\sqrt{\pi})\exp(-p^2w^2/2\hbar^2), \tag{18}$$

where g has been specialized to a Gaussian weight function, and $|\psi_\perp(t)\rangle$ was introduced in equation (15). In equation (18), we have introduced the transverse width w of the electronic

wave packet (electron beam). The following is the physical situation to which the state given by equation (18) may be associated. Let us imagine that an electron is injected into a cavity at time $t = 0$ through a small hole of width w. On the basis of our earlier study of the true initial value problem (Bergou and Varró 1981a), we expect that the sudden coupling of the electron with the (highly occupied) cavity mode results, *in essence*, in the formation of the state $|\psi\rangle$ defined by equation (18). In the present paper, we restrict our analysis to the study of the spatio-temporal evolution of these approximate states (which are entangled already at $t = 0$). Owing to the unitarity of the displacement operator D in equation (15), the superposition $|\psi\rangle$ defined by equation (18) is a normalized state in the product space of the photon–electron system. In order to have an explicit form of this state, we express it in the electron's coordinate representation and, at the same time, expand it in terms of the photon number eigenstates

$$|\Xi(\vec{r}, t)\rangle \equiv \sum_{k=-n_0}^{\infty} |n_0+k\rangle\langle n_0+k|\langle\vec{r}|\psi\rangle, \qquad \int \mathrm{d}^2 r \langle\Xi(\vec{r}, t)|\Xi(\vec{r}, t)\rangle = 1. \tag{19}$$

The summation index in the above equation has been shifted merely for the sake of convenience later in the text. The normalization condition in equation (19) follows from the proper normalization $\langle\psi|\psi\rangle = 1$ and from the completeness relations

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = 1_{\text{photon}}, \qquad \int \mathrm{d}^2 r |\vec{r}\rangle\langle\vec{r}| = 1_{\text{electron}},$$

where 1_{photon} and 1_{electron} denote the unit operators on the Hilbert spaces H_{photon} and H_{electron} of the quantized mode and of the electron, respectively. The scalar products in the first equation of equation (19) can be expressed as

$$\langle n_0+k|\langle\vec{r}|\psi\rangle = \int_0^{\infty} \mathrm{d}p\, pg(p) \frac{1}{2\pi\hbar} \exp\left(-\frac{\mathrm{i}}{\hbar}\frac{p^2}{2m_\perp}t\right) \int_0^{2\pi} \mathrm{d}\chi \exp\left[\frac{\mathrm{i}}{\hbar} pr\cos(\chi-\varphi)\right]$$
$$\times \langle n_0+k|D[\sigma(\vec{p})]|n_0\rangle. \tag{20}$$

In equations (19) and (20), r and φ denote the radial and angular transverse positions of the electron, respectively, i.e. $\vec{r} = r(\cos\varphi, \sin\varphi)$. The physical meaning of the matrix elements given by equation (20) is that they are joint probability amplitudes of the simultaneous detection of an electron (at position $\vec{r}$ and instance of time t) and of a definite number of photons $n_0 + k$. As is shown in appendix A, for large values of n_0, an asymptotic expression can be calculated for the matrix elements of the displacement operator, equation (16) (see equations leading to equation (A.15)). We note that the integrals over the electron's momentum in equation (20) can be evaluated exactly for an arbitrary (not necessarily a large) value of n_0 (see the *exact analytic expression* in equation (A.7)), but hereinafter, we shall only discuss cases of large n_0 values, and use the approximation stemming from equation (A.15). After the integration with respect to the azimuth angle χ in momentum space, we obtain

$$\langle n_0+k|\langle\vec{r}|\psi\rangle = (w/\hbar\sqrt{\pi})(-\mathrm{i})^k \mathrm{e}^{-\mathrm{i}k\varphi} \int_0^{\infty} \mathrm{d}p\, p \exp(-p^2w^2/2\hbar^2) \exp\left(-\frac{\mathrm{i}}{\hbar}\frac{p^2}{2m_\perp}t\right)$$
$$\times J_k\left(\sqrt{2}\frac{eA_0}{mc\hbar\Omega}p\right) J_k(pr/\hbar) + O(n_0^{-3/4}). \tag{21}$$

In equation (21), we have introduced the quantity $A_0 = (c/\omega)\sqrt{2\pi\rho\hbar\omega}$, which is formally equal to the amplitude of the classical vector potential $\vec{A}_{\rm cl} = A_0(\vec{\varepsilon}\,{\rm e}^{-{\rm i}\omega t} + \vec{\varepsilon}^{\,*}\,{\rm e}^{{\rm i}\omega t})$ associated with the photon density $\rho = n_0/L^3$, if we make the identification $u = E_{\rm cl}^2/4\pi = \rho\hbar\omega$. Here, u denotes the energy density of the mode, with $\vec{E}_{\rm cl} = -\partial\vec{A}_{\rm cl}/\partial ct = (\omega/c)\sqrt{2}A_0(\vec{\varepsilon}_x \sin\omega t - \vec{\varepsilon}_y \cos\omega t)$ being the electric field strength. According to equation (A.18), we obtain from equation (21) the limit form in the case of high initial occupation numbers,

$$\langle n_0 + k|\langle\vec{r}|\psi\rangle = \frac{1}{w\sqrt{\pi}}\frac{(-{\rm i})^k{\rm e}^{-{\rm i}k\varphi}}{(1+{\rm i}t/\tau)}\exp\left[-\frac{(\mu\Lambda/w)^2+(r/w)^2}{2(1+{\rm i}t/\tau)}\right]$$

$$\times I_k\left[\frac{(\mu\Lambda/w)\cdot(r/w)}{(1+{\rm i}t/\tau)}\right] + O(n_0^{-3/4}), \tag{22}$$

where I_k is a modified Bessel function of the first kind of the order k, and

$$\begin{aligned}\mu &\equiv \frac{{\rm e}A_0\sqrt{2}}{mc^2} \equiv \frac{{\rm e}F_0}{mc\omega} = 10^{-9}\sqrt{I}/E_{\rm ph},\\ 1/\tau &\equiv \hbar/m_\perp w^2 \approx \hbar/mw^2,\\ \Lambda &\equiv c/\Omega \approx \lambda\!\!\!^{-} \equiv \lambda/2\pi.\end{aligned} \tag{23}$$

In equation (23), we have defined the 'dimensionless intensity parameter' μ, whose numerical value can be expressed in terms of the intensity I of the mode of the radiation field measured in W cm^{-2}, and of the photon energy $E_{\rm ph}$ measured in electron volts. We have also introduced the amplitude of the electric field strength $F_0 \equiv (\omega/c)A_0\sqrt{2} = \sqrt{4\pi\rho\hbar\omega}$ and the wavelength λ of the radiation. The approximate equalities in equation (23) are valid for large L. If we let both n_0 and L going to infinity, in such a way that the photon density is a fixed parameter, then the last term on the right-hand side of equation (22) can be suppressed, and μ can formally be associated with a classical electric field of amplitude F_0. Then in equation (22) $\mu\Lambda \to \mu\lambda/2\pi$ becomes just the amplitude of oscillation of a classical electron under the action of the electric field of the radiation $\vec{E} = F_0(\vec{\varepsilon}_x \sin\omega t - \vec{\varepsilon}_y \cos\omega t)$. This can easily be shown by solving the Newton equations $m\ddot{x} = -{\rm e}E_x$ and $m\ddot{y} = -{\rm e}E_y$. Thus, the dimensionless quantity $\mu\Lambda/w \approx \mu\lambda/2\pi w$ is the ratio of the amplitude of the classical oscillation of the electron to the initial transverse width at $t = 0$ of the electron packet (electron beam). We emphasize that the above remarks were made simply to outline a rough picture in order to give a physical background of the parameters introduced in equations (21) and (23). Of course, we are not saying that a classical electric field can be associated with an even very highly occupied number state. This can consistently be done by using the Schrödinger–Glauber coherent states (Glauber 1963a, 1963b). Anyway, our preliminary investigations on this latter subject clearly show that parameters of a similar sort naturally appear there, too; thus, these parameters are allowed to be used in realistic numerical estimates. The timescale parameter τ defined in equation (23) can be related to the period $T = 2\pi/\omega$ of the radiation field through the 'bare timescale parameter' $\tau_0 \equiv mw^2/\hbar$,

$$\tau = (m_\perp/m)\tau_0 = \left[(1+\omega_{\rm p}^2/2\omega^2)/(1-\omega_{\rm p}^2/2\omega^2)\right]\tau_0, \quad \omega\tau_0 = (1/2)(2mc^2/\hbar\omega)(2\pi w/\lambda)^2. \tag{24}$$

The 'transverse mass' $m_\perp$, defined in equation (13), can *in principle* be much larger than the 'bare mass' m, if $\omega_{\rm p}^2/2\omega^2$ approaches (from below) 1. Consequently, the transverse spreading of the electronic wave packet can *in principle* be reduced due to the interaction with the electromagnetic radiation. From equation (19), by neglecting the term of order $n_0^{-3/4}$ in

equation (22), we have the following approximate form for $|\Xi(\vec{r},t)\rangle$:

$$|\Xi(\vec{r},t)\rangle \to |\tilde{\Xi}(\vec{r},t)\rangle \equiv \psi_g(r,t)\sum_{k=-n_0}^{\infty}(-\mathrm{i})^k \mathrm{e}^{-\mathrm{i}k\varphi} I_k[\gamma(r,t)]\cdot|n_0+k\rangle, \tag{25}$$

where

$$\begin{aligned}\psi_g(r,t) &\equiv \frac{1}{w\sqrt{\pi}}\frac{1}{(1+\mathrm{i}t/\tau)}\exp\left[-\frac{(\mu\Lambda/w)^2+(r/w)^2}{2(1+\mathrm{i}t/\tau)}\right],\\ \gamma(r,t) &\equiv \frac{(\mu\Lambda/w)\cdot(r/w)}{(1+\mathrm{i}t/\tau)}.\end{aligned} \tag{26}$$

It can be proved by explicit calculation (see the derivation of equation (A.22)) that in the limit $n_0\to\infty$ (and $L\to\infty$, but $n_0/L^3=\rho$ fixed), the approximate states $|\tilde{\Xi}(\vec{r},t)\rangle$, given by equation (25), are also properly normalized, like the exact states in equations (19). By using the index transformation $n_0+k=n$, we obtain an alternative form of equation (25),

$$|\tilde{\Xi}(\vec{r},t)\rangle = \psi_g(r,t)(-i)^{-n_0}\,\mathrm{e}^{\mathrm{i}n_0\varphi}\sum_{n=0}^{\infty}(-i)^n\,\mathrm{e}^{-\mathrm{i}n\varphi} I_{n-n_0}[\gamma(r,t)]\cdot|n\rangle. \tag{27}$$

Apart from the factors $\mathrm{e}^{-\mathrm{i}n\varphi}$, for $t=0$, when $\gamma(r,t)$ is real, the 'photon part' (the sum with respect to n) on the right-hand side of equation (27) has the same functional form as the 'number-phase minimum uncertainty states' $|\Psi\rangle$, equation (6), derived by Jackiw (1968). Notice that the quantum number n_0 (corresponding to the parameter ν in Jackiw's solution) is an integer number in our case, in contrast to ν, which always has to have a non-vanishing fractional part. The other difference is that the normalization constant κ in equation (6) is determined by the equation

$$|\kappa|^2\sum_{n=0}^{\infty} I^2_{n-\nu}(\gamma)=1,$$

but in our case,

$$\|\,\tilde{\Xi}(\vec{r},t)\,\|^2 = |\psi_g(r,t)|^2\sum_{n=0}^{\infty}|I_{n-n_0}[\gamma(r,t)]|^2 \neq 1$$

(where $\|\cdot\|$ means the norm in the Hilbert subspace of the quantized mode). For the 'photon part' of the state in equation (27), a normalization similar to that of Jackiw's states can be achieved by using

$$|\kappa'(r,t)|^2\sum_{n=0}^{\infty}|I_{n-n_0}[\gamma(r,t)]|^2=1.$$

At the end of the present section, we give some numerical illustrations of the spatio-temporal behaviour of the joint probabilities $|\langle n_0+k|\langle\vec{r}|\psi\rangle|^2$ on the basis of the analytic expression equation (22) found in the large photon excitation limit. These are the probabilities of those simultaneous events when the electron is detected at position $\vec{r}$, and k photons are absorbed or emitted at some position (which need not necessarily be the same as that of the electron; rather, for practical reasons, it should be different). In the numerical examples, we will always assume that the wavelength of the quantized electromagnetic radiation is of the order of

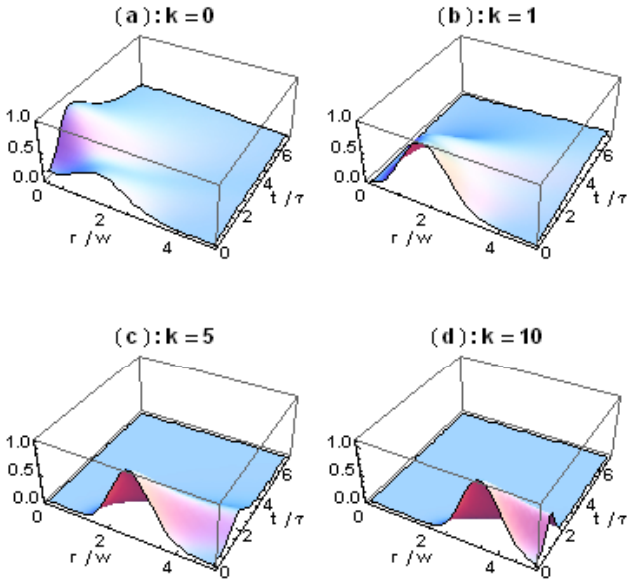

Figure 1. Spatio-temporal distribution of the joint probability coming from equation (22) or (25) for different values of the number of emitted excess photons k. In each figure, we have used a numerical normalization factor, in order for the maximum values of the vertical coordinates to be roughly unity. These factors are the following: (a) 30 for $k = 0$, (b) 90 for $k = 1$, (c) 2×10^4 for $k = 5$ and (d) 8×10^8 for $k = 25$.

$\lambda \approx 10^{-4}$ cm, i.e. the photon energy is of the order of $\hbar\omega \approx 1$ eV. In this case, the dimensionless intensity parameter μ, introduced in equations (23), is simply expressed as $\mu = 10^{-9} I^{1/2}$, where I denotes the intensity of the photon field divided by 1 W cm^{-2}. Besides, we shall also assume that the wavelength parameter Λ, introduced in equations (23), to a good approximation, coincides with $\lambda/2\pi$. This means, according to the definition of Ω in equations (9), that the one-electron plasma frequency ω_p is assumed to be much smaller than ω, the frequency of the optical field. In figure 1, we show the spatio-temporal distribution of the joint probabilities $|\langle n_0 + k|\langle \vec{r}|\psi\rangle|^2$ for some given k-values.

Because of the symmetry of the modified Bessel functions with respect to the change of the sign of their order, $k \to -k$, the same distributions result for photon absorptions. The surfaces in figure 1 illustrate the electron's detection probability at the radial position r and at the instance of time t, if we know for certain that k photons have been emitted or absorbed (detected by a spatially separated counter). Here, we have taken $(\mu\Lambda/w) = 2$, which corresponds to an intensity 10^{12} W cm^{-2}. This can be seen from equations (23) by assuming that $\lambda/w = 4\pi \times 10^3$. For an optical field, λ is of the order of 10^{-4} cm; accordingly, w is of the order of 10^{-8} cm. As is seen, in the case of the initial intensity we have considered, the elastic channel ($k = 0$) and the one-photon channels ($k = \pm 1$) dominate, and the higher order channels ($|k| > 1$) have much lower joint probabilities. As is seen in figure 1, for $t = 0$ the maxima of the dominant low-order joint probabilities ($k = 0, \pm 1$) show up at the normalized radial position $\sim \mu\lambda/2\pi w = 2$, which is just the ratio of the amplitude of the electron oscillation to

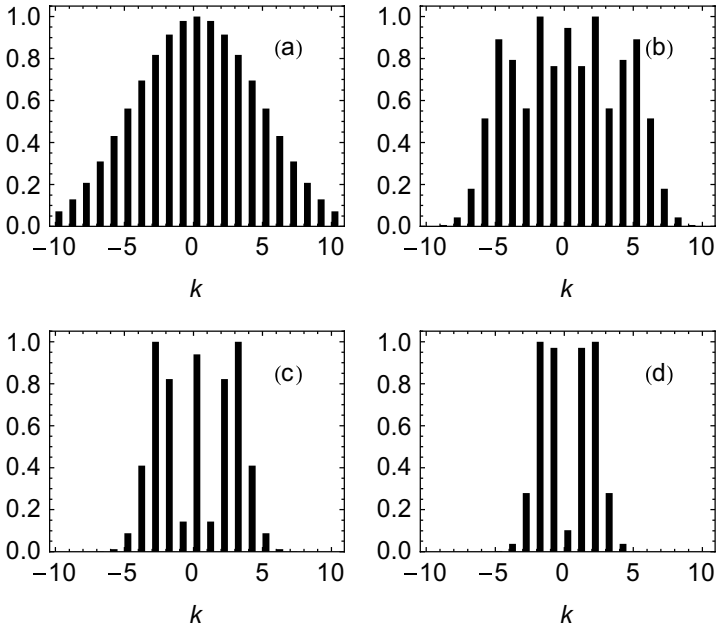

Figure 2. The excess photon number distribution around the central large initial photon number n_0 for different ratios of t/τ and r/w. This means that the k-dependences along the lines $(t/\tau) = s \cdot (r/w)$ on the r–t-plane are plotted for different s-values. The tangents are: $s = 0.3$ in (a), $s = 0.6$ in (b), $s = 1$ in (c) and $s = 1.5$ in (d).

the spatial width of the electronic wave packet. This behaviour can be explained on the basis of the functional form of the position representation of the entangled photon–electron state given by equations (25) and (26).

Our next example, figure 2, illustrates the (joint) photon distribution $|\langle n_0 + k|\langle \vec{r}|\psi\rangle|^2$ around the central large initial photon number n_0 for different ratios of t/τ and r/w, i.e. now the spatio-temporal position of the electron detection is a given parameter in each figure. Here, $(\mu\Lambda/w) = 4$ is assumed, and $r/w = 10$. The probabilities are normalized to their maximum values, which are 1.10×10^{-5} in (a), 9.15×10^{-5} in (b), 1.97×10^{-4} in (c) and 1.86×10^{-4} in (d). As the tangent s varies from 0.3 to 1.5, the distribution undergoes a qualitative change. The monotonic distribution illustrated by (a) goes over to oscillating distributions, as is shown by (b), (c) and (d).

From figure 2, we can conclude that in certain regions on the r–t-plane (where the electron is being detected) the probability distributions of the simultaneous detection of k photons have qualitatively different shapes. It is clear from the functional form of these probabilities, deduced from equations (25) and (26), that in figure 1(a) we see a '*modified* Bessel function behaviour', and on the other hand, in figures 1(b)–(d) we encounter '*ordinary* Bessel function behaviour'. In the first case, the distribution has a similar form as the set $\{I_k^2(x)\}$, where x is a real number. In the last three cases, the distributions have a similar form as $\{J_k^2(x)\}$ for different real values of x, and these distributions 'oscillate', i.e. there appear local minima and maxima as k varies. We have numerically studied the shapes of the (joint) photon number distributions $|\langle n_0 + k|\langle \vec{r}|\psi\rangle|^2$, and located three regions of the r–t-plane, where the shapes of the distributions are qualitatively different. The result in a special case is displayed in figure 3.

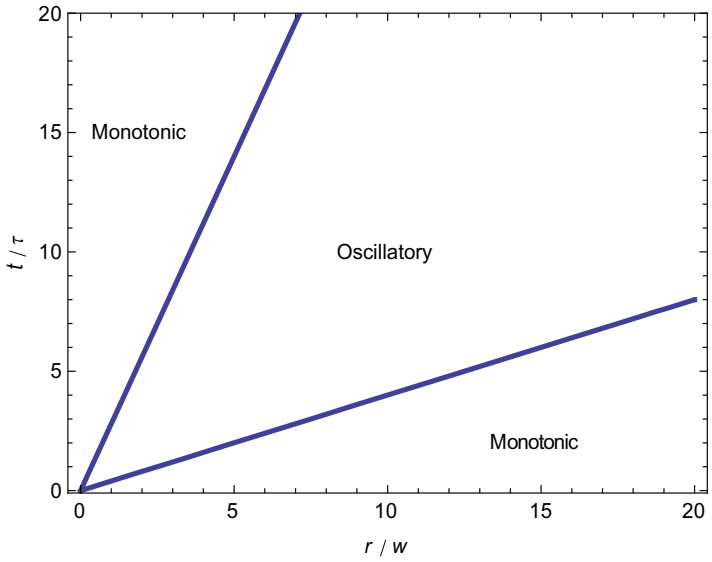

Figure 3. Schematic representation of the space–time regions, where the shapes of the joint probability distributions are qualitatively different.

As in figure 2, we assume $(\mu\Lambda/w) = 4$, which corresponds to an intensity $2 \times 10^{12}\,\mathrm{W\,cm^{-2}}$, and $\lambda/w = 4\pi \times 10^3$. The tangents of the lower lines and the upper lines are 0.4 and 2.8, respectively. In this case, if an electron detection takes place in the spatio-temporal ranges $(t/\tau) < 0.4 \times (r/w)$ or $(t/\tau) > 2.8 \times (r/w)$, then the photon number distributions would be one-peaked 'monotonic' distributions, like in figure 2(a). On the other hand, in the range defined by the relations $(t/\tau) > 0.4 \times (r/w)$ and $(t/\tau) < 2.8 \times (r/w)$, the photon number distributions are 'oscillatory', i.e. the joint probability distributions have several local minima and maxima. Of course, the transition from the monotonic regime to the oscillatory regime is not as sharp as the figure would suggest at first glance.

5. Reduced density operators and entanglement entropies

Let us first calculate the density operator $\hat{P}$ of the quantized mode associated with the entangled state $|\psi\rangle$ introduced in equation (18). By taking the partial trace (denoted below by Tr') of the dyad $|\psi\rangle\langle\psi|$ with respect to the electron variables, we have

$$\begin{aligned}\hat{P} \equiv \mathrm{Tr}'\{|\psi\rangle\langle\psi|\} &= \int \mathrm{d}^2 p' \langle \vec{p}'|\psi\rangle\langle\psi\,\vec{p}'\rangle \\ &= \sum_{k=-n_0}^{\infty} \sum_{l=-n_0}^{\infty} |n_0+k\rangle\langle n_0+l| \int \mathrm{d}^2 p |g(\vec{p})|^2 \langle n_0+k|D[\sigma(\vec{p})]|n_0\rangle \\ &\quad \times \{\langle n_0+l|D[\sigma(\vec{p})]|n_0\rangle\}^*. \end{aligned} \tag{28}$$

In obtaining equation (28), the orthogonality of the transverse momentum eigenstates has been used, $\langle\vec{p}|\vec{p}'\rangle = \delta_2(\vec{p}-\vec{p}')$. As is shown in appendix B, the integral on the right-hand side of equation (28) can be analytically evaluated, yielding the *exact photon number distribution* given by equation (B.3). In the following, we shall not discuss this general distribution, but rather, we

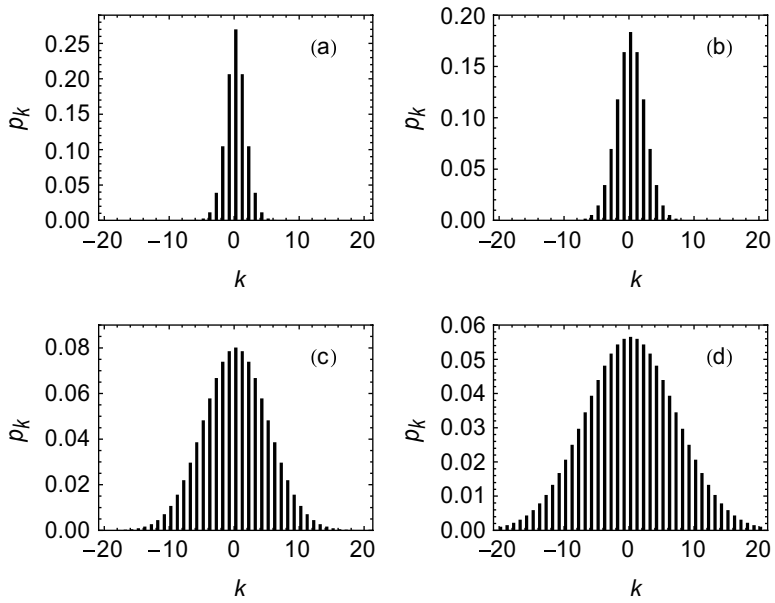

Figure 4. True photon number distribution $\{p_k\}$ (derived from the reduced density operator, and given by equation (29)) for four q (intensity) values, namely for $q = 2.5$ in (a), $q = 5$ in (b), $q = 25$ in (c) and $q = 50$ in (d).

shall study the case of high initial photon excitations. For large values of n_0, the reduced density operator $\hat{P}$ can be brought to the form (see the derivation leading to equation (B.7))

$$\hat{P} = \sum_{k=-n_0}^{\infty} |n_0 + k\rangle\, p_k \langle n_0 + k| + O(n_0^{-3/4}), \qquad p_k \equiv I_k(q)\mathrm{e}^{-q}, \tag{29}$$

where

$$q \equiv (1/2)(\mu\Lambda/w)^2 \approx (1/2)\mu^2(\lambda/2\pi w)^2, \tag{30}$$

and the quantities μ and Λ have already been defined in equations (23). As is proved in appendix B, the set of weights $\{p_k\}$ is properly normalized, i.e. $\sum_{k=-\infty}^{\infty} p_k = 1$. Owing to the property $I_{-k}(z) = I_k(z)$, the distribution given in equations (29) is symmetric to $k = 0$, which means that the weights of k-photon absorptions are the same as that of k-photon emissions. We note that, in fact, the set $\{p_k\}$ governs the *true photon number distribution*, rather than the expansion coefficients of $|\Xi(\vec{r}, t)\rangle$ (obtained from equation (22), and used in equations (25) and (27)). These latter expansion coefficients are *joint probability amplitudes* of detecting an electron at a position $\vec{r}$, at an instance of time t and, at the same time, detecting $n_0 + k$ photons.

If we assume $\lambda/w = 4\pi \times 10^3$, like in figure 1, then these q values used in obtaining figure 4 correspond to intensities of 1.25×10^{12}, 2.5×10^{12}, 1.25×10^{13} and 2.5×10^{13} W cm^{-2}, respectively. The terminology 'true photon number distribution' we have used for $\{p_k\}$ can be justified by the fact that this set is built up from the (diagonal) elements of the density operator of the photon field, equation (29), which, of course, does not contain electron variables, since these latter ones have been traced out. In figure 4, it is clearly seen that as the intensity is increasing, the higher order absorption and induced emission events become more and more dominant, and the widths of the distributions become larger and larger. Not an unexpected result! Let us note

that the results based on our present analysis do not contradict the famous statement according to which 'a free electron cannot absorb or emit a photon'. This statement, which can be found in any of the basic texts on QED, relies on the perturbation theory of the S-matrix approach dealing with asymptotic incoming and outgoing plane waves representing the electrons and the photons. The interaction of the electron with a strong laser *beam* is, in fact, a many-body interaction, in the sense that the beam can be considered as a superposition of plane electromagnetic waves propagating in different directions, and taking part in high-order induced processes. This question has long been discussed e.g. by Bergou *et al* (1983), who used a relativistic semiclassical description. The study of such more general problems is out of the scope of the present paper. Here, we have used a very simplified scheme (non-relativistic description of the electron, restriction to one-mode interactions, dipole approximation, which, on the other hand, are well justified in the range of parameters taken in our numerical examples below). Our goal here is merely to show some basic characteristics of the entangled photon–electron systems.

The von Neumann entropy, $S_{\rm photon}$, associated with the distribution $\{p_k\}$, can be considered as one of the natural measures of the degree of entanglement of the photon–electron system. By using equation (29), we obtain

$$\begin{aligned} S_{\rm photon}[\hat{P}] &\equiv -{\rm Tr}[\hat{P}\log\hat{P}] \\ &= S_{\rm photon}[\{p_k\}] \equiv -\sum_{k=-\infty}^{\infty} p_k \log p_k \\ &= q - \left\{ I_0(q)\log[I_0(q)] + 2\sum_{k=1}^{\infty} I_k(q)\log[I_k(q)] \right\} \exp(-q), \end{aligned} \tag{31}$$

where q has been defined in equation (30). According to equation (31), the entropy of the quantized radiation field does not depend on time. This is because the entangled photon–electron state introduced in equation (18) in a sense is a *stationary state*, though it contains explicitly the time variable in a complicated manner, as is shown by its analytic form given by equations (25) and (26). The state $|\psi\rangle$, equation (18), is *not* a solution of a true initial-value problem where we would have assumed an initially non-interacting system (represented by a product state) and switch on the interaction at $t = 0$ somehow. We leave the study of this latter problem for a separate work in progress. In figure 5, we illustrate the intensity dependence of the von Neumann entropy of the photon field. In the parameter range we have considered, the entropy curve, shown in figure 5 by using log–linear scale, becomes a straight line after the intensity has passed the value $\sim 10^{12}\,{\rm W\,cm^{-2}}$. This means that the entropy $S_{\rm photon}[\{p_k\}]$ increases logarithmically with the intensity. At zero intensity, the entropy vanishes because the interaction of the photon and the electron is negligible in this case (since the photon density is zero).

In obtaining figure 5, we have assumed that the independent variable q in $S_{\rm photon}[\{p_k\}]$ is expressed numerically as $q = 2 \times [I/({\rm W\,cm^{-2}})]$. This means, according to equation (30), that $\lambda/w = 4\pi \times 10^3$ is assumed, i.e. the wavelength of the radiation field is roughly 10 000 times larger than the initial transverse size of the electronic wave packet. For an optical field, we have $\lambda \sim 10^{-4}$ cm, so in the case we have considered, $w \sim 10^{-8}$ cm.

Now let us derive the reduced density operator $\hat{P}_{\rm e}$ of the electron, associated with the entangled state $|\psi\rangle$, which has been introduced in equation (18). By taking equation (15) into

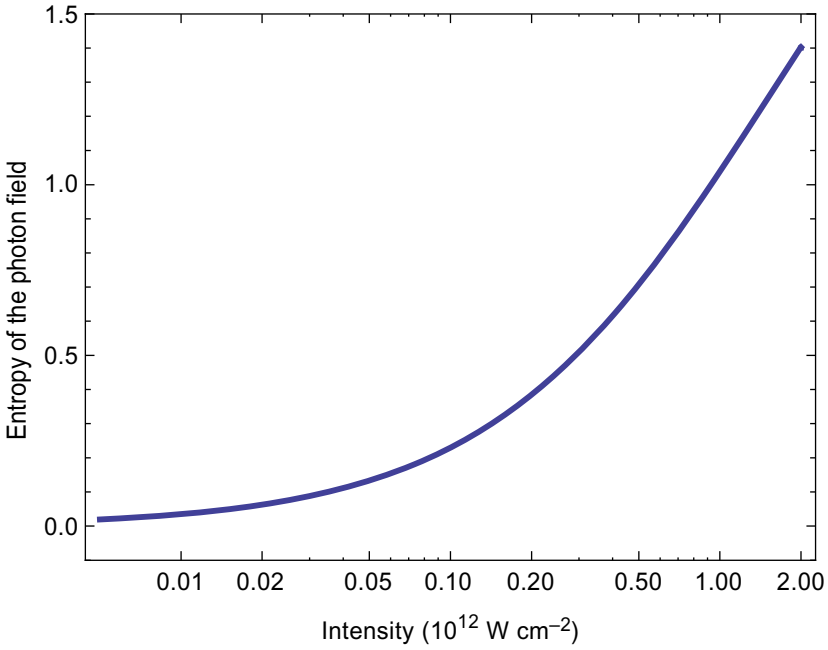

Figure 5. The intensity dependence of the von Neumann entropy of the photon field defined by equation (31).

account, the partial trace (denoted by Tr″) of $|\psi\rangle\langle\psi|$ with respect to the photon variables reads

$$\begin{aligned}\hat{P}_{\mathrm{e}} &= \mathrm{Tr}''\{|\psi\rangle\langle\psi|\} \equiv \sum_{n=0}^{\infty}\langle n|\psi\rangle\langle\psi|n\rangle \\ &= \int \mathrm{d}^2p' \int \mathrm{d}^2p'' g(\vec{p}')g^*(\vec{p}'')|\vec{p}'\rangle\langle\vec{p}''|\exp[-\mathrm{i}t(p'^2 - p''^2)/2m_\perp\hbar] \\ &\quad \times \langle n_0|D^+[\sigma(\vec{p}'')]D[\sigma(\vec{p}')]|n_0\rangle. \end{aligned} \tag{32}$$

The matrix elements of $\hat{P}_{\mathrm{e}}$ *in momentum space* can be calculated by using equations (B.10) and (16), yielding

$$\begin{aligned}P_{\mathrm{e}}(\vec{p}', \vec{p}'') \equiv \langle\vec{p}'|\hat{P}_{\mathrm{e}}|\vec{p}''\rangle &= g(\vec{p}')g^*(\vec{p}'')\exp[-\mathrm{i}t(p'^2 - p''^2)/2m_\perp\hbar]\exp\{\mathrm{iIm}[\sigma(\vec{p}')\sigma^*(\vec{p}'')]\} \\ &\quad \times L_{n_0}(|\sigma(\vec{p}' - \vec{p}'')|^2)\exp(-|\sigma(\vec{p}' - \vec{p}'')|^2/2), \end{aligned} \tag{33}$$

where $L_n(x)$ denotes Laguerre polynomials of the order n.

The diagonal matrix elements of $\hat{P}_{\mathrm{e}}$ in momentum space are simply given by the modulus squared of the weight function $g(\vec{p})$ defined in equation (18), i.e.

$$P_{\mathrm{e}}(\vec{p}, \vec{p}) = |g(\vec{p})|^2 = (w/\hbar)^2\Pi(\vec{k}), \qquad \Pi(\vec{k}) \equiv (1/\pi)\exp(-k^2), \qquad \vec{k} \equiv (w/\hbar)\vec{p}. \tag{34}$$

In equations (34), we have introduced the dimensionless momentum variable $\vec{k}$ and the density function $\Pi(\vec{k})$. According to equations (B.14) and (B.15), the matrix elements of the reduced density operator $\hat{P}_{\mathrm{e}}$ *in position space* can be expressed as scalar products of the position representation of the entangled photon–electron states introduced in equation (19),

$$P_{\mathrm{e}}(\vec{r}, \vec{r}'; t) = \langle\Xi(\vec{r}', t)|\Xi(\vec{r}, t)\rangle = \langle\tilde{\Xi}(\vec{r}', t)|\tilde{\Xi}(\vec{r}, t)\rangle + O(n_0^{-3/4}), \tag{35}$$

where $|\tilde{\Xi}(\vec{r},t)\rangle$ has been defined in equations (25) and (26). As is shown in appendix B, in cases of very high photon excitations (more accurately, in the limit $n_0 \to \infty$), the density function in equation (35) becomes

$$P_{\rm e}(\vec{r},\vec{r}';t) \to \langle\tilde{\Xi}(\vec{r}',t)|\tilde{\Xi}(\vec{r},t)\rangle = (1/w^2)F_{\rm e}(\vec{x},\vec{x}';t), \qquad \text{with } \vec{x}\equiv\vec{r}/w, \quad \vec{x}'\equiv\vec{r}'/w,$$

$$F_{\rm e}(\vec{x},\vec{x}';t) \equiv \frac{1}{\pi(1+t^2/\tau^2)}\exp\left[-\frac{(\mu\Lambda/w)^2}{(1+t^2/\tau^2)}\right] \times \exp\left[-\frac{x'^2+x^2+(it/\tau)(x'^2-x^2)}{2(1+t^2/\tau^2)}\right] \times I_0\left[(\mu\Lambda/w)\frac{x'+x+(it/\tau)(x'-x)}{(1+t^2/\tau^2)}\right]. \tag{36}$$

The diagonal matrix elements of the electron's reduced density operator are determined by the dimensionless density function, which we call the *true position distribution of the electron*, since the photon variables have been traced out. We obtain

$$P(\vec{x},t) \equiv F_{\rm e}(\vec{x},\vec{x};t) = \frac{1}{\pi(1+t^2/\tau^2)}\exp\left[-\frac{(\mu\Lambda/w)^2+x^2}{(1+t^2/\tau^2)}\right] \times I_0\left[2\frac{(\mu\Lambda/w)\cdot x}{(1+t^2/\tau^2)}\right]. \tag{37}$$

The distribution $P(\vec{x},t)$ is normalized to unity for any instance of time. This can be shown by using a similar procedure as that applied in the proof of equation (A.22).

According to equation (29), the density operator of the photon field is diagonal; thus, we were able to write down immediately the explicit formula in equation (31) for the von Neumann entropy. As is seen from equations (33) and (36), the electron's density operator $\hat{P}_{\rm e}$, equation (32), neither in momentum representation nor in position representation is diagonal. In order to calculate the von Neumann entropy of the electron, first we have to diagonalize $\hat{P}_{\rm e}$, which, at the moment, seems to us a hopeless task. In order to avoid this difficulty we rather study the so-called *linear entropy* H, which has a close connection with the second-order *Rényi entropy*. The definition of H reads

$$H \equiv 1 - {\rm Tr}\hat{\rho}^2 = \exp(H_2)+1, \qquad H_2 \equiv -\log {\rm Tr}\hat{\rho}^2 \tag{38}$$

where H_2 is the second-order Rényi entropy, and $\hat{\rho}$ is some generic density operator. The linear entropy has been used by several authors (see e.g. Zurek *et al* 1993 and Joos *et al* 2003), because it is much easier to calculate (since we do not need the diagonalization of $\hat{\rho}$), and, on the other hand, it is a good alternative to the von Neumann entropy as a measure of entanglement. Really, H vanishes for a pure state, and it is maximum when the eigenvalues of $\hat{\rho}$ are identical (which is the case of maximum mixing). Another useful quantity to characterize the entanglement in a two-particle system is the Schmidt number K (see Nielsen and Chuang 2000) whose definition is $K \equiv [{\rm Tr}(\hat{\rho}^2)]^{-1}$, where $\hat{\rho}$ denotes the reduced density operator of either of the two particles. The Schmidt number has been extensively used to characterize continuous-variable entanglement by Fedorov and co-workers (see Fedorov *et al* 2004, 2005, 2006, 2007) in their thorough analyses of the wave packet dynamics in breakup processes, like ionization of atoms and dissociation of molecules (see, in particular, Fedorov *et al* 2006, where a unifying overview of rapidly separating systems is presented).

Let us first calculate the linear entropy of the photon field associated with the distribution $\{p_k\}$ given by equation (29). The details of the calculation can be found in appendix B. According to equation (B.21), we have

$$\begin{aligned} H_{\text{photon}}[\hat{P}] &\equiv 1 - \text{Tr}\hat{P}^2 \\ &= H_{\text{photon}}[\{p_k\}] \equiv 1 - \sum_{k=-\infty}^{\infty} p_k^2 \\ &= 1 - \text{e}^{-2q} \sum_{k=-\infty}^{\infty} I_k^2(q) = 1 - I_0(2q)\text{e}^{-2q}. \end{aligned} \tag{39}$$

In order to calculate the linear entropy of the electron, we first need an explicit expression of $\hat{P}_{\text{e}}^2$, which can be obtained from equation (B.9) by a straightforward calculation,

$$\begin{aligned} \hat{P}_{\text{e}}^2 = \int \text{d}^2 p_1 \int \text{d}^2 p_2 \int \text{d}^2 p_3 g(\vec{p}_1)|g(\vec{p}_2)|^2 g^*(\vec{p}_3) \exp[-\text{i}t(p_1^2 - p_3^2)/2m_\perp \hbar] \\ \times \langle n_0|D^+[\sigma(\vec{p}_2)]D[\sigma(\vec{p}_1)]|n_0\rangle \cdot \langle n_0|D^+[\sigma(\vec{p}_3)]D[\sigma(\vec{p}_2)]|n_0\rangle \cdot |\vec{p}_1\rangle\langle\vec{p}_3|. \end{aligned} \tag{40}$$

The trace of $\hat{P}_{\text{e}}^2$ can be calculated analytically; thus, we can derive an exact expression for the linear entropy of the electron, as is shown in appendix B by equation (B.28). In the limit $n_0 \to \infty$ (and at the same time the quantization volume $L^3 \to \infty$, such that the photon density n_0/L^3 is being kept fixed), according to equation (B.29), we obtain

$$\begin{aligned} &\text{Tr}\hat{P}_{\text{e}}^2 = \int_0^{\infty} \text{d}x\, x J_0^2[(\mu\Lambda/w)x] \exp(-x^2/2) = I_0(2q)\text{e}^{-2q}, \\ &q \equiv \tfrac{1}{2}(\mu\Lambda/w)^2. \end{aligned} \tag{41}$$

By using equation (41) and the general definition given by equation (38), the linear entropy of the electron becomes

$$H_{\text{electron}}[\hat{P}_{\text{e}}] = 1 - I_0(2q)\text{e}^{-2q}, \tag{42}$$

which coincides with the linear entropy of the photon field given by equation (39),

$$H_{\text{electron}}[\hat{P}_{\text{e}}] = H_{\text{photon}}[\hat{P}] = 1 - I_0(2q)\text{e}^{-2q}. \tag{43}$$

Equation (43) expresses a remarkable consistency in our calculations leading to the analytic results given by equations (39) and (42). Regardless of using the discrete photon number distribution $\{p_k\}$ given by equation (29), or using the double integral of the dyads $|\vec{p}'\rangle\langle\vec{p}''|$, parametrized by the two (continuous) momentum variables of the electron in equation (32), we end up with the same result for the entanglement entropies. After all, the identity of the two entropies *must* be required for an entangled system consisting of *two* subsystems. In figure 6, we compare the intensity dependences of the von Neumann entropy $S_{\text{photon}}[\hat{P}]$ and of the (identical) linear entropies $H_{\text{electron}}[\hat{P}_{\text{e}}] = H_{\text{photon}}[\hat{P}]$ of the electron and of the quantized mode given by equations (31), (42) and (39), respectively. We plotted the curves by using log–linear scale in a larger intensity range than considered in figure 5. As in figure 5, we have assumed that the independent variable q is expressed numerically as $q = 2 \times [I/(\text{W cm}^{-2})]$, i.e. we have taken $\lambda/w = 4\pi \times 10^3$. The logarithmic increase of the von Neumann entropy with the intensity (curve 'S') is clearly seen in figure 5. The linear entropy (curve 'H') is always smaller than the

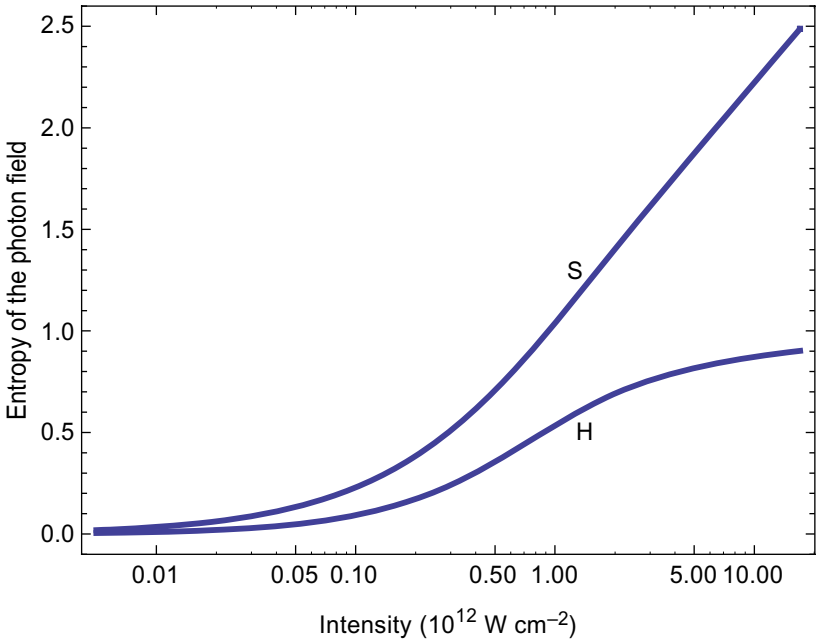

Figure 6. A comparison of the intensity dependences of the von Neumann entropy $S_{\text{photon}}[\hat{P}]$ and of the (identical) linear entropies $H_{\text{electron}}[\hat{P}_{\text{e}}] = H_{\text{photon}}[\hat{P}]$ given by equations (31), (42) and (39), respectively.

von Neumann entropy, and increases much more slowly than the latter one. The increase of each measure of the entanglement by increasing the intensity is after all not an unexpected result, since the interaction of the photons and the free electron is becoming stronger and stronger as the photon density is getting larger.

6. Summary

In the present paper, we have discussed entanglement between photons and electrons. We have shown that the entangled photon–electron states introduced by us have a close connection with the critical states introduced by Jackiw (1968), which minimize a number-phase uncertainty product of the photon field. These states are of essentially the same mathematical structure as that of Jackiw's states, and naturally appear in the non-perturbative analysis of the simplest interaction of QED we have considered, namely the interaction of a free electron with a quantized mode of the electromagnetic radiation. On the basis of our analysis, we have given a simple interpretation of states of Jackiw's type; thus, we may say that these latter states have a physical significance, rather than being mere mathematical constructions, as they were originally thought of. Besides, we have derived exact analytic expressions for the reduced density operators of the photon field and of the free electron, and determined the von Neumann entropy of the photon, and the linear entropy of the photon and of the electron.

In section 1 we gave a brief historical overview of the development of the concepts on entanglement and on the related earliest basic experiments. Moreover, we sketched the most important approaches to the problem of the quantal phase of the linear oscillator (or of a quantized mode of the radiation field). On purpose, we quoted the earlier references too, so that the interested reader can keep track of the evolution of the concepts on the quantal

phase from the very beginning. In section 2, we have summarized the basic properties of the cosine and sine operators of the quantal phase introduced by Susskind and Glogower (1964), and we have presented the critical states constructed by Jackiw (1968), which minimize the uncertainty product of the number operator and the cosine operator. On the basis of our earlier work Bergou and Varró (1981a), in section 3, we determined the exact stationary states of the interacting photon–electron system. These states are simple product states whose photon parts are generalized coherent states, and the electron parts are momentum eigenstates. Section 4 has been devoted to the construction of the entangled photon–electron states which are defined as Gaussian superpositions (with respect to the electron's momentum variable) of the stationary states as discussed in section 3. As we have already emphasized, these entangled states defined by equation (18) are not solutions of a true initial-value problem where we would have assumed an initially non-interacting system (represented by a *bare* product state) and switch on the interaction, say, at $t = 0$ somehow. We leave the study of this latter problem for a separate work in progress. In appendix A, we have given an exact analytic expression for the expansion coefficients of the entangled states (with respect to the number state basis of the photon's Hilbert space and in position representation in the electron's Hilbert space), and in the main text, we studied the properties of the associated probability distributions for various parameter values in the large excitation limit. The expansion coefficients of the entangled states, obtained from equation (22), and used in equations (25) and (27), are in fact joint probability amplitudes of detecting an electron at some position and at an instance of time and, at the same time, detecting a certain definite number of photons. The basic features of these joint probabilities have been illustrated in figures 1–3. In section 5, we presented the reduced density operators of the photon field and of the free electron, and with the help of them the true photon number distribution and the electron's momentum and position distributions have been calculated. The exact expressions have been derived in appendix B, and in the main text, we have analysed the characteristics of these distributions in the large excitation limit. As measures of the entanglement, the von Neumann entropy of the photon field and the linear entropies of the photon field and of the electron have also been calculated exactly, and closed analytic forms for them were given in the large excitation limit. We have proved by an explicit calculation that the latter two quantities coincide. Our results are displayed in figures 4–6, which show the true photon number distribution, the intensity dependence of the von Neumann entropy of the photon field and the comparison of the intensity dependence of the linear entropy and of the von Neumann entropy, respectively. Finally, we note that it may seem to be a serious restriction to confine our (non-perturbative) study to the analysis of interactions of a free electron with only *one* quantized mode of the radiation field. In reality, of course, the electron interacts with the *whole assembly* of the modes due to e.g. secondary spontaneous emission processes (see, for instance, the case of the Compton scattering). The study of the interactions only with one mode can be justified if this mode is in a very highly excited state (as has been mostly assumed in the present paper). In this special case (which, on the other hand, is of great importance in the physics of nonlinear processes taking place in laser–matter interactions), the interactions with the other modes (or with some other third body) can be treated as small perturbations. In this context, see e.g. the works of Bergou and Varró (1981a, 1981b). In order to have an estimate for the magnitude of the excitation degree n_0 in realistic laser systems, we can use equation (B.34) of appendix B, which gives a numerical formula for the mean photon occupation number. In table B.1, we have summarized the numerical values of the parameters we are interested in, for three kinds of laser radiation. It is seen that for intensities manageable nowadays, the mean occupation number can

be enormously large. Of course, a c-number electric field strength cannot be associated with even a very highly occupied number state in the strict sense. This association can consistently be done, for instance, by using the coherent states of Schrödinger–Glauber type (Glauber 1963a, 1963b). We plan to present the study of the coherent superpositions of the entangled photon–electron states elsewhere.

Acknowledgments

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Appendix A. Derivation of the explicit form of the entangled photon-electron states

Here, we show the basic steps leading to the exact analytic form of the matrix elements, equation (20),

$$\langle n_0+k|\langle\vec{r}|\psi\rangle = \int_0^\infty \mathrm{d}ppg(p)\frac{1}{2\pi\hbar}\exp\left(-\frac{\mathrm{i}}{\hbar}\frac{p^2}{2m_\perp}t\right)\int_0^{2\pi}\mathrm{d}\chi\exp\left[\frac{\mathrm{i}}{\hbar}pr\cos(\chi-\varphi)\right] \times\langle n_0+k|D[\sigma(\vec{p})]|n_0\rangle, \tag{A.1}$$

and we derive the approximate form, equation (22), of them. We prove that the asymptotic states $|\tilde{\Xi}(\vec{r},t)\rangle$, equation (25), are properly normalized.

In order to start with, first we give an alternative form of the expansion coefficients of the generalized coherent states, displayed in equation (16),

$$c_{n_0+k,n_0} = \langle n_0+k|D[\sigma(\vec{p})]|n_0\rangle = \begin{cases}[n_0!/(n_0+|k|)!]^{1/2}\sigma^{|k|}L_{n_0}^{|k|}(|\sigma|^2)\mathrm{e}^{-|\sigma|^2/2}, & (k>0),\\ [(n_0-|k|)!/n_0!]^{1/2}(-\sigma^*)^{|k|}L_{n_0-|k|}^{|k|}(|\sigma|^2)\mathrm{e}^{-|\sigma|^2/2}, & (n_0\leqslant k<0),\end{cases} \tag{A.2}$$

where, according to equation (10),

$$\sigma(\vec{p}) = -(\mathrm{e}a/mc\hbar\Omega)\vec{p}\cdot\vec{\varepsilon}^* = (-\mathrm{e}^{-\mathrm{i}\chi})bx, \qquad \text{with } b\equiv\frac{\mathrm{e}a}{mc\hbar\Omega\sqrt{2}}\frac{\hbar}{w} \quad \text{and} \quad x\equiv pw/\hbar. \tag{A.3}$$

The quantities a, Ω and w have been introduced in equations (8), (10) and (18), and the dimensionless variable $x = pw/\hbar$ has been used to calculate the radial integral in equation (A.1). The integration with respect to the azimuth angle χ in momentum space in equation (A.1) can be carried out by using the Jacobi–Anger formula for the electron-plane wave

$$\exp\left[\frac{\mathrm{i}}{\hbar}pr\cos(\chi-\varphi)\right] = \sum_{l=-\infty}^{\infty}\mathrm{i}^l J_l(pr/\hbar)\mathrm{e}^{\mathrm{i}l(\chi-\varphi)} \tag{A.4}$$

(see Gradshteyn and Ryzhik (2000), formula 8.551.4), and the elementary relation $\int_0^{2\pi} \mathrm{d}\chi \mathrm{e}^{\mathrm{i}n\chi} = 2\pi\delta_{n,0}$. Then, from equations (A.1) to (A.4), we obtain

$$\langle n_0+k|\langle\vec{r}|\psi\rangle = (1/w\sqrt{\pi})(-i)^k \mathrm{e}^{-\mathrm{i}k\varphi} b^s \int_0^\infty \mathrm{d}x x^{s+1} \mathrm{e}^{-\beta\cdot x^2} J_s(yx) \times \begin{cases} [n_0!/(n_0+s)!]^{1/2} L_{n_0}^s(b^2x^2), & (k>0), \\ [(n_0-s)!/n_0!]^{1/2} L_{n_0-s}^s(b^2x^2), & (-n_0 \leqslant k < 0), \end{cases} \tag{A.5}$$

where we have introduced the notations

$$s \equiv |k|, \qquad y \equiv r/w \qquad \text{and} \qquad \beta \equiv \frac{1}{2}\left(1+\mathrm{i}\frac{t}{\tau}+b^2\right), \quad \text{with } \tau \equiv \frac{m_\perp w^2}{h}. \tag{A.6}$$

$J_s(z)$ denotes ordinary Bessel function of the first kind of the order s. The radial integral in equation (A.5) can be expressed in an analytic form by using the formula 7.421.4 of Gradshteyn and Ryzhik (2000), yielding

$$\langle n_0+k|\langle\vec{r}|\psi\rangle = \frac{(-\mathrm{i})^k \mathrm{e}^{-\mathrm{i}k\varphi}}{w\sqrt{\pi}2\beta}\left(\frac{by}{2\beta}\right)^s \exp\left(-\frac{y^2}{4\beta}\right) \times \begin{cases} \left[\frac{n_0!}{(n_0+s)!}\right]^{1/2}\left(1-\frac{b^2}{\beta}\right)^{n_0} L_{n_0}^s\left[\frac{b^2y^2}{4\beta(b^2-\beta)}\right], & (k>0) \\ \left[\frac{(n_0-s)!}{n_0!}\right]^{1/2}\left(1-\frac{b^2}{\beta}\right)^{n_0-s} L_{n_0-s}^s\left[\frac{b^2y^2}{4\beta(b^2-\beta)}\right], & (-n_0 \leqslant k < 0). \end{cases} \tag{A.7}$$

In order to study the asymptotic behaviour of the above exact expression for large n_0 values, we use the limit formula (Erdélyi 1953, formula 10.12(36))

$$\lim_{n\to\infty}[n^{-s}L_n^s(z/n)] = z^{-s/2}J_s(2z^{1/2}) \tag{A.8}$$

for given s- and z-values. Moreover, we take the limit in such a way that, though both n_0 and the quantization volume L^3 are going to infinity, the photon density $\rho \equiv n_0/L^3$ is considered as a fixed parameter. According to the definitions of b and β given by equations (A.3) and (A.6), respectively, we have

$$b = \left(\frac{\mathrm{e}A_0\sqrt{2}}{mc^2}\right)\cdot\left(\frac{\Lambda}{w}\right)\cdot\frac{1}{2\sqrt{n_0}} = \frac{1}{\sqrt{n_0}}\frac{1}{2}(\mu\Lambda/w), \quad \text{where} \quad \Lambda = \frac{c}{\Omega} \to \frac{\lambda}{2\pi},$$
$$A_0 = \left(\frac{c}{\omega}\right)\sqrt{2\pi\left(\frac{n_0}{L^3}\right)\hbar\omega}, \qquad \mu = \frac{\mathrm{e}A_0\sqrt{2}}{mc^2} \qquad \text{and} \qquad \beta \to \frac{1}{2}\left(1+\mathrm{i}\frac{t}{\tau}\right). \tag{A.9}$$

(The physical meaning of the amplitude A_0 and the dimensionless intensity parameter μ is discussed in section 4 of the main text.) The argument of the Laguerre polynomials and the power expression in front then read, respectively

$$-\frac{b^2y^2}{4\beta^2} = -\left[\frac{(\mu\Lambda/w)(r/w)}{2(1+\mathrm{i}t/\tau)}\right]^2\frac{1}{n_0} \equiv \frac{z}{n_0},$$
$$\left(\frac{by}{2\beta}\right)^s = \frac{1}{\mathrm{i}^s}\frac{z^{s/2}}{n_0^{s/2}}, \tag{A.10}$$

where the definition of y in equation (A.6) has also been taken into account. Thus, on the basis of equations (A.8)–(A.10), in the first line on the right-hand side of equation (A.7), we have

$$\left(\frac{by}{2\beta}\right)^s \left[\frac{n_0!}{(n_0+s)!}\right]^{1/2} L_{n_0}^s\left[-\frac{b^2y^2}{4\beta^2}\right]$$

$$= [(1+1/n_0)\cdot(1+2/n_0)\cdots(1+s/n_0)]^{-1} \times (1/\mathrm{i}^s) z^{s/2} n_0^{-s} L_{n_0}^s(z/n_0) \to (1/\mathrm{i}^s) J_s(2z^{1/2})$$

$$= I_s\left[\frac{(\mu\Lambda/w)(r/w)}{(1+\mathrm{i}t/\tau)}\right], \tag{A.11}$$

where $I_s(\zeta)$ is a modified Bessel function of the first kind of the order s. In obtaining the final result in equation (A.11), we have used the relation $J_s(\mathrm{i}\zeta) = \mathrm{i}^s I_s(\zeta)$, valid for integer s and for an arbitrary complex number ζ (see Gradshteyn and Ryzhik (2000), formula 8.406.3). The same expression, as in equation (A.11), comes out from the second line on the right-hand side of equation (A.7). The factor $(1-b^2/\beta)^{n_0}$ can be written down in the following alternative form:

$$(1-b^2/\beta)^{n_0} = \left[1-\frac{(\mu\Lambda/w)^2}{2(1+\mathrm{i}t/\tau)}\frac{1}{n_0}\right]^{n_0} \to \exp\left[-\frac{(\mu\Lambda/w)^2}{2(1+\mathrm{i}t/\tau)}\right], \tag{A.12}$$

where the well-known relation $\lim_{n\to\infty}(1+z/n)^n = \mathrm{e}^z$ has been used. From equations (A.7), (A.11) and (A.12), by taking the exponential $\exp(-y^2/4\beta)$ also into account, finally, we obtain in the large n_0 limit

$$\langle n_0+k|\langle\vec{r}|\psi\rangle = \frac{1}{w\sqrt{\pi}}\frac{(-\mathrm{i})^k \mathrm{e}^{-\mathrm{i}k\varphi}}{(1+\mathrm{i}t/\tau)} \exp\left[-\frac{(\mu\Lambda/w)^2+(r/w)^2}{2(1+\mathrm{i}t/\tau)}\right]$$
$$\times I_k\left[\frac{(\mu\Lambda/w)\cdot(r/w)}{(1+\mathrm{i}t/\tau)}\right] \quad (n_0\to\infty). \tag{A.13}$$

It is possible to give an alternative (and shorter) derivation of the result, equation (A.13), by starting with an asymptotic formula already for the matrix elements equation (A.2), instead of using the exact integral equation (A.7). In order to do that, let us apply the following formula of Hilb's type (Erdélyi (1953), formula 10.15(2)):

$$\mathrm{e}^{-x/2}x^{s/2}L_n^s(x) = \frac{\Gamma(n+s+1)}{(\nu/4)^{s/2}n!}J_s[(\nu x)^{1/2}] + O(n^{s/2-3/4}) \tag{A.14}$$

valid for $s>-1$ uniformly in $0<x\leqslant K<\infty$, where $\nu\equiv 4n+2s+2$. By applying equation (A.14), we have

$$\langle n_0+k|D[\sigma(\vec{p})]|n_0\rangle = (-\mathrm{e}^{-\mathrm{i}\chi})^k J_k(2\sqrt{n_0}|\sigma(\vec{p})|) + O(n_0^{-3/4}). \tag{A.15}$$

The integration with respect to the azimuth angle χ in momentum space in equation (A.1) can again be easily carried out by using the Jacobi–Anger formula, equation (A.4). Thus, on the basis of equation (A.15), we obtain

$$\langle n_0+k|\langle\vec{r}|\psi\rangle = (w/\hbar\sqrt{\pi})(-\mathrm{i})^k\mathrm{e}^{-\mathrm{i}k\varphi}\int_0^\infty \mathrm{d}pp\exp(-p^2w^2/2\hbar^2)\exp\left(-\frac{\mathrm{i}}{\hbar}\frac{p^2}{2m_\perp}t\right)$$
$$\times J_k\left(\sqrt{2}\frac{\mathrm{e}A_0}{mc\hbar\Omega}p\right)J_k(pr/\hbar) + O(n_0^{-3/4}). \tag{A.16}$$

By using Weber's second exponential integral (Watson (1944), formula 13.31(1))

$$\int_0^\infty \mathrm{d}x\, x J_k(ax) J_k(bx) \mathrm{e}^{-c^2x^2} = (1/2c^2)\exp[-(a^2+b^2)/4c^2] I_k(ab/2c^2), \qquad (A.17)$$

(where the condition $|\arg c| < \pi/4$ is to be satisfied), finally, we obtain

$$\langle n_0+k|\langle \vec{r}|\psi\rangle = \frac{1}{w\sqrt{\pi}} \frac{(-\mathrm{i})^k \mathrm{e}^{-\mathrm{i}k\varphi}}{(1+\mathrm{i}t/\tau)} \exp\left[-\frac{(\mu\Lambda/w)^2+(r/w)^2}{2(1+\mathrm{i}t/\tau)}\right] \cdot I_k\left[\frac{(\mu\Lambda/w)\cdot(r/w)}{(1+\mathrm{i}t/\tau)}\right] + O(n_0^{-3/4}), \qquad (A.18)$$

which is equivalent to the expression given in equation (A.13).

At the end of the present appendix, we prove by a direct calculation that the approximate entangled photon–electron states $|\tilde{\Xi}(\vec{r},t)\rangle$, defined in equation (25) of section 4 in the main text, is properly normalized in the limit $n_0 \to \infty$. Really, after the φ-integration, we have

$$\int \mathrm{d}^2 r \langle \tilde{\Xi}(\vec{r},t)|\tilde{\Xi}(\vec{r},t)\rangle = \frac{2}{w^2(1+t^2/\tau^2)} \exp\left[-\frac{(\mu\Lambda/w)^2}{(1+t^2/\tau^2)}\right] \times \int_0^\infty \mathrm{d}r r \sum_{k=-\infty}^{\infty} I_k[\gamma^*(r,t)]\, I_k[\gamma(r,t)] \exp\left[-\frac{(r/w)^2}{(1+t^2/\tau^2)}\right]. \qquad (A.19)$$

Now, owing to the formulae 8.406.3 and 8.538.1 of Gradshteyn and Ryzhik (2000),

$$I_k(z) = \mathrm{i}^{-k} J_k(\mathrm{i}z) \qquad \text{and} \qquad \sum_{k=-\infty}^{\infty} (-1)^k J_k(\xi)\, J_k(\xi) = J_0(2\xi), \qquad (A.20)$$

respectively, we obtain

$$\sum_{k=-\infty}^{\infty} I_k(\gamma^*) I_k(\gamma) = J_0[\mathrm{i}(\gamma^*+\gamma)] = J_0\left[2\mathrm{i}\frac{(\mu\Lambda/w)}{\sqrt{1+t^2/\tau^2}} \cdot \frac{(r/w)}{\sqrt{1+t^2/\tau^2}}\right] \qquad (A.21)$$

where we have also taken into account the definition of $\gamma(r,t)$ given by equation (26). On the basis of equations (A.19) and (A.21), the normalization integral becomes

$$\int \mathrm{d}^2 r \langle \tilde{\Xi}(\vec{r},t)|\tilde{\Xi}(\vec{r},t)\rangle = 2\exp\left[-\frac{(\mu\Lambda/w)^2}{(1+t^2/\tau^2)}\right] \times \int_0^\infty \mathrm{d}x\, x J_0\left[2\mathrm{i}\frac{(\mu\Lambda/w)}{\sqrt{1+t^2/\tau^2}} \cdot x\right] \exp(-x^2) = 1, \qquad \forall t; \qquad (A.22)$$

thus, the proper normalization 'survives' the approximation in the limit $n_0 \to \infty$. In obtaining the result, equation (A.22), we have used Weber's first exponential integral

$$\int_0^\infty \mathrm{d}x\, x J_0(ax)\, \mathrm{e}^{-c^2x^2} = \frac{1}{2c^2} \exp\left(-\frac{a^2}{4c^2}\right) \qquad (A.23)$$

where $|\arg c| < \pi/4$, and a is an arbitrary complex number (Watson (1944), formula 13.3(1)).

Appendix B. Derivation of the reduced density operators and of the entanglement entropies

In the present appendix, first we calculate the density operator $\hat{P}$ of the quantized mode associated with the entangled state $|\psi\rangle$ introduced in equation (18). According to equation (28) of section 4, by taking the partial trace (denoted below by Tr′) of the dyad $|\psi\rangle\langle\psi|$ with respect to the electron variables, we have

$$\hat{P} \equiv \mathrm{Tr}'\{|\psi\rangle\langle\psi|\} = \int \mathrm{d}^2 p' \langle \vec{p}'|\psi\rangle\langle\psi|\vec{p}'\rangle = \sum_{k=-n_0}^{\infty}\sum_{l=-n_0}^{\infty} |n_0+k\rangle\langle n_0+l| \int \mathrm{d}^2 p |g(\vec{p})|^2 \times \langle n_0+k|D[\sigma(\vec{p})]|n_0\rangle\{\langle n_0+l|D[\sigma(\vec{p})]|n_0\rangle\}^*. \tag{B.1}$$

By using equations (A.2) and (A.3), after having performed the integration with respect to the azimuth angle χ, from equation (B.1) we obtain

$$\hat{P} = \sum_{k=-n_0}^{\infty} |n_0+k\rangle\langle n_0+k| b^{2s} \int_0^{\infty} \mathrm{d}x x^s \exp[-(1+b^2)x] \times \begin{cases} \dfrac{n_0!}{(n_0+s)!}[L_{n_0}^s(b^2x)]^2, & (k>0), \\ \dfrac{(n_0-s)!}{n_0!}[L_{n_0-s}^s(b^2x)]^2, & (-n_0 \leqslant k < 0), \end{cases} \tag{B.2}$$

where the definitions of b in equation (A.3), and the notation $s \equiv |k|$ have been used. The integrals in equation (B.2) can be analytically done on the basis of formula 7.414.13 of Gradshteyn and Ryzhik (2000),

$$\hat{P} = \sum_{k=-n_0}^{\infty} |n_0+k\rangle P_k \langle n_0+k|,$$
$$P_k = \frac{(b^2)^{|k|}}{(1+b^2)^{1+|k|}} \begin{cases} P_{n_0}^{(|k|,0)}[(1+b^4)/(1-b^4)], & (k>0), \\ P_{n_0-|k|}^{(|k|,0)}[(1+b^4)/(1-b^4)], & (-n_0 \leqslant k < 0), \end{cases} \tag{B.3}$$

where $P_n^{(\alpha,\beta)}(x)$ are Jacobi polynomials of the order n. From equation (B.1), it is clear that because of the unitarity of the displacement operator $D[\sigma(\vec{p})]$ and since the profile function $g(p)$ is normalized, the distibution $\{P_k\}$ defined in equation (B.3) is normalized to unity. Because $|(1+b^4)/(1-\mathrm{b}^4)| > 1$ for both $b > 1$ and $0 < b < 1$, the argument of the Jacobi polynomials in the above equation is out of the interval $[-1, +1]$, in the interior of which all the zeros are located. Thus, the weights P_k cannot take on the zero value. Moreover, out of the interval $[-1, +1]$ these polynomials are all positive (which, by the way, is required by a true probability distribution). In the limit $n_0 \to \infty$, the weights P_k can be approximated with the help of the asymptotic formula (see Erdélyi (1953), formula 10.8(41))

$$\lim_{n\to\infty} n^{-\alpha} P_n^{(\alpha,\beta)}(1 - z^2/2n^2) = (z/2)^{-\alpha} J_\alpha(z), \tag{B.4}$$

where in our case $n = n_0$ or $n = n_0 - s$, $\alpha = s \equiv |k|$, $\beta = 0$, and from equation (A.9)

$$b^2 = (\mu\Lambda/w)^2/2n_0, \qquad z = iq, \qquad q \equiv (1/2)(\mu\Lambda/w)^2. \tag{B.5}$$

With the help of equations (B.4) and (B.5), the weights in equation (B.3) can be approximately expressed in terms of the modified Bessel functions $P_k \approx \mathrm{i}^{-|k|} J_{|k|}(\mathrm{i}q) = I_k(q)$. However, this approximate distribution is not normalized properly; it only gives the relative photon occupation probabilities. In order to derive a properly normalized approximate distribution in the large n_0 limit, directly from equation (B.2), we are now proceeding differently as before. By using Hilb's formula, equation (A.14), the integral in equation (B.2) can be asymptotically expressed as

$$\int_0^\infty \mathrm{d}x\, J_k^2[(\mu\Lambda/w)\sqrt{x}]\mathrm{e}^{-x} = 2\int_0^\infty \mathrm{d}x\, x J_k^2[(\mu\Lambda/w)x]\mathrm{e}^{-x^2} = I_k(q)\mathrm{e}^{-q}, \tag{B.6}$$

where we have used Weber's second exponential integral given by equation (A.17). Thus, for large values of n_0, the reduced density operator $\hat{P}$ can be brought to the form

$$\begin{aligned} \hat{P} &= \sum_{k=-n_0}^{\infty} |n_0+k\rangle p_k \langle n_0+k| + O(n_0^{-3/4}), \\ p_k &\equiv I_k[(1/2)(\mu\Lambda/w)^2]\exp[-(1/2)(\mu\Lambda/w)^2] = I_k(q)\mathrm{e}^{-q}, \end{aligned} \tag{B.7}$$

where $q \equiv (1/2)(\mu\Lambda/w)^2 \approx (1/2)\mu^2(\lambda/2\pi w)^2$, and the quantities μ and Λ have already been defined in equation (A.9) (and in equation (23) of section 4 in the main text). By using the Jacobi–Anger formula, equation (A.4), and the relation $I_n(z) = \mathrm{i}^{-n} J_n(\mathrm{i}z)$, it can easily be proved that $\sum_{n=-\infty}^{\infty} I_n(z) = \mathrm{e}^z$; hence the set of weights $\{p_k\}$ is properly normalized, i.e. $\sum_{k=-\infty}^{\infty} p_k = 1$. The von Neumann entropy, S_{photon} associated with the distribution $\{p_k\}$ defined in equation (29), can be considered as one of the natural measures of the degree of entanglement in the photon–electron system. By using equation (B.7), we obtain

$$S_{\mathrm{photon}}[\{p_k\}] = -\sum_{k=-\infty}^{\infty} p_k \log p_k = q - \left\{I_0(q)\log[I_0(q)] + 2\sum_{k=1}^{\infty} I_k(q)\log[I_k(q)]\right\}\exp(-q), \tag{B.8}$$

where the parameter q has been defined after equation (B.7).

Now let us derive the reduced density operator $\hat{P}_{\mathrm{e}}$ of the electron, associated with the entangled state $|\psi\rangle$ introduced in equation (18). By taking equation (15) into account, the partial trace (denoted by Tr'') of $|\psi\rangle\langle\psi|$ with respect to the photon variables reads

$$\begin{aligned} \hat{P}_{\mathrm{e}} = \mathrm{Tr}''\{|\psi\rangle\langle\psi|\} &\equiv \sum_{n=0}^{\infty}\langle n|\psi\rangle\langle\psi|n\rangle \\ &= \int \mathrm{d}^2p' \int \mathrm{d}^2p'' g(\vec{p}')g^*(\vec{p}'')|\vec{p}'\rangle\langle\vec{p}''|\exp[-\mathrm{i}t(p'^2 - p''^2)/2m_\perp\hbar] \\ &\times\langle n_0|D^+[\sigma(\vec{p}'')]D[\sigma(\vec{p}')]|n_0\rangle. \end{aligned} \tag{B.9}$$

The matrix elements of $\hat{P}_{\mathrm{e}}$ in momentum space can be calculated by using the relation

$$D^+[\sigma'']D[\sigma'] = D[\sigma' - \sigma'']\exp[\mathrm{i}\,\mathrm{Im}(\sigma''^*\sigma')], \tag{B.10}$$

and equation (16),

$$\begin{aligned} P_{\mathrm{e}}(\vec{p}', \vec{p}'') \equiv \langle\vec{p}'|\hat{P}_{\mathrm{e}}|\vec{p}''\rangle &= g(\vec{p}')g^*(\vec{p}'')\exp[-\mathrm{i}t(p'^2 - p''^2)/2m_\perp\hbar]\exp\{\mathrm{i}\,\mathrm{Im}[\sigma(\vec{p}')\sigma^*(\vec{p}'')]\} \\ &\times L_{n_0}(|\sigma(\vec{p}' - \vec{p}'')|^2)\exp(-|\sigma(\vec{p}' - \vec{p}'')|^2/2), \end{aligned} \tag{B.11}$$

where $L_n(x)$ are the Laguerre polynomials of the order n.

From equation (B.11), it is clear that diagonal matrix elements of $\hat{P}_e$ in momentum space are simply given by the modulus squared of the weight function $g(\vec{p})$ defined in equation (18), i.e.

$$P_e(\vec{p},\vec{p}) = |g(\vec{p})|^2 = (w/\hbar)^2\Pi(\vec{k}), \qquad \forall t, \tag{B.12}$$

where we have introduced the dimensionless momentum distribution $\Pi(\vec{k})$,

$$\Pi(\vec{k}) \equiv (1/\pi)\exp(-k^2), \qquad \vec{k} \equiv (w/\hbar)\vec{p}. \tag{B.13}$$

The matrix elements of the reduced density operator $\hat{P}_e$ in position space can be determined by using the identities,

$$\begin{aligned} P_e(\vec{r},\vec{r}';t) &\equiv \langle\vec{r}|\mathrm{Tr}''\{|\psi\rangle\langle\psi|\}|\vec{r}'\rangle = \sum_{k=-n_0}^{\infty}\langle\vec{r}|\langle n_0+k|\psi\rangle\langle\psi|n_0+k\rangle|\vec{r}'\rangle \\ &= \sum_{k=-n_0}^{\infty}\sum_{k'=-n_0}^{\infty}\langle\psi|\vec{r}'\rangle|n_0+k'\rangle\langle n_0+k'|\cdot|n_0+k\rangle\langle n_0+k|\vec{r}|\psi\rangle. \end{aligned} \tag{B.14}$$

By comparing the factors on the right-hand side of equation (B.14) with the coordinate representation of the entangled photon–electron state, defined in equation (19), we realize that, in fact, the matrix elements are expressed by the following scalar products:

$$P_e(\vec{r},\vec{r}';t) = \langle\Xi(\vec{r}',t)\Xi(\vec{r},t)\rangle = \langle\tilde{\Xi}(\vec{r}',t)|\tilde{\Xi}(\vec{r},t)\rangle + O(n_0^{-3/4}), \tag{B.15}$$

where $|\tilde{\Xi}(\vec{r},t)\rangle$ has been defined in equations (25) and (26). In the case of very high photon excitations (more accurately, in the limit $n_0 \to \infty$) the density function in equation (B.15) becomes

$$P_e(\vec{r},\vec{r}';t) \to \langle\tilde{\Xi}(\vec{r}',t)|\tilde{\Xi}(\vec{r},t)\rangle = (1/w^2)F_e(\vec{x},\vec{x}';t), \quad \text{with } \vec{x} \equiv \vec{r}/w, \quad \vec{x}' \equiv \vec{r}'/w,$$

and

$$\begin{aligned} F_e(\vec{x},\vec{x}';t) \equiv{}& \frac{1}{\pi(1+t^2/\tau^2)}\exp\left[-\frac{(\mu\Lambda/w)^2}{(1+t^2/\tau^2)}\right] \\ &\times\exp\left[-\frac{x'^2+x^2+(\mathrm{i}t/\tau)(x'^2-x^2)}{2(1+t^2/\tau^2)}\right]\times I_0\left[(\mu\Lambda/w)\frac{x'+x+(\mathrm{i}t/\tau)(x'-x)}{(1+t^2/\tau^2)}\right]. \end{aligned} \tag{B.16}$$

In obtaining equation (B.16), we have used the formulae 8.406.3 and 8.538.1 of Gradshteyn and Ryzhik (2000), which were also used in the derivation of equation (A.21),

$$I_k(z) = \mathrm{i}^{-k}J_k(\mathrm{i}z), \qquad \sum_{k=-\infty}^{\infty}(-1)^k J_k(z_1)J_k(z_2) = J_k(z_1+z_2), \tag{B.17}$$

and the explicit form of $\gamma(r,t)$ given by equation (26). The diagonal matrix elements of the electron's reduced density operator are determined by the dimensionless density function

$$P(\vec{x},t) \equiv F_e(\vec{x},\vec{x};t) = \frac{1}{\pi(1+t^2/\tau^2)}\exp\left[-\frac{(\mu\Lambda/w)^2+x^2}{(1+t^2/\tau^2)}\right]\times I_0\left[2\frac{(\mu\Lambda/w)\cdot x}{(1+t^2/\tau^2)}\right]. \tag{B.18}$$

The density function $P(\vec{x}, t)$ is normalized to unity for any instance of time, as can be shown similarly to the proof of equation (A.22).

According to equation (B.7), the density operator of the photon field turned out to be diagonal; thus, we were able to write down immediately the explicit formula in equation (B.8) for the von Neumann entropy,

$$S_{\text{photon}}[\hat{P}] \equiv -\text{Tr}[\hat{P}\log\hat{P}] = S[\{p_k\}]. \tag{B.19}$$

As we see from equations (B.11) and (B.16), the electron's density operator $\hat{P}_e$, equation (B.9), neither in momentum representation nor in position representation is diagonal. In order to calculate the von Neumann entropy of the electron, first we have to diagonalize $\hat{P}_e$, which we have not been able to do up to now. In order to avoid this difficulty, here, we study the so-called *linear entropy* H, which has a close connection with the second-order *Rényi entropy*. The definition of H reads

$$\begin{aligned} H &\equiv 1 - \text{Tr}\hat{\rho}^2 = \exp(H_2) + 1, \\ H_2 &\equiv -\log Tr\,\hat{\rho}^2, \end{aligned} \tag{B.20}$$

where H_2 is the second-order Rényi entropy, and $\hat{\rho}$ is some generic density operator.

Let us first calculate the linear entropy of the photon field associated with the distribution $\{p_k\}$ given by (B.7).

$$\begin{aligned} H_{\text{photon}}[\hat{P}] &\equiv 1 - \text{Tr}\hat{P}^2 \\ &= H_{\text{photon}}[\{p_k\}] = 1 - \sum_{k=-\infty}^{\infty} p_k^2 \\ &= 1 - \text{e}^{-2q}\sum_{k=-\infty}^{\infty} I_k^2(q) = 1 - I_0(2q)\text{e}^{-2q}, \end{aligned} \tag{B.21}$$

where q has been defined in equation (B.5). In deriving equation (B.21), we have used the summation theorem of the Bessel functions already given in equation (A.20).

In order to calculate the linear entropy of the electron, we need first an explicit expression of $\hat{P}_e^2$, which can be obtained from equation (B.9) by a straightforward calculation,

$$\begin{aligned} \hat{P}_e^2 = \int \text{d}^2p_1 \int \text{d}^2p_2 \int \text{d}^2p_3 g(\vec{p}_1)|g(\vec{p}_2)|^2 g^*(\vec{p}_3)\exp[-\text{i}t(p_1^2 - p_3^2)/2m_\perp\hbar] \\ \times\langle n_0|D^+[\sigma(\vec{p}_2)]D[\sigma(\vec{p}_1)]|n_0\rangle\cdot\langle n_0|D^+[\sigma(\vec{p}_3)]D[\sigma(\vec{p}_2)]|n_0\rangle\cdot|\vec{p}_1\rangle\langle\vec{p}_3|. \end{aligned} \tag{B.22}$$

The trace of $\hat{P}_e^2$ is the expressed as

$$\text{Tr}\hat{P}_e^2 = \int \text{d}^2p_1 \int \text{d}^2p_2 |g(\vec{p}_1)|^2\cdot|g(\vec{p}_2)|^2\cdot|\langle n_0|D^+[\sigma(\vec{p}_2)]D[\sigma(\vec{p}_1)]|n_0\rangle|^2. \tag{B.23}$$

By using equation (B.10) and (A.2), the above expression can be brought to the form

$$\text{Tr}\hat{P}_e^2 = \int \text{d}^2p_1 \int \text{d}^2p_2 |g(\vec{p}_1)|^2\cdot|g(\vec{p}_2)|^2\left[L_{n_0}(|\sigma(\vec{p}_1 - \vec{p}_2)|^2)\right]^2\exp(-|\sigma(\vec{p}_1 - \vec{p}_2)|^2). \tag{B.24}$$

With the help of the following transformation of the integration variables:

$$\vec{p}' \equiv \vec{p}_1 - \vec{p}_2, \quad \vec{p}'' \equiv \vec{p}_1 + \vec{p}_2, \quad \text{with} \left|\frac{\partial(p_{1x}, p_{1y}, p_{2x}, p_{2y})}{\partial(p'_x, p'_y, p''_x, p''_y)}\right| = \frac{1}{4}, \tag{B.25}$$

the integrations with respect to $\vec{p}'$ and $\vec{p}''$ can be separated,

$$\mathrm{Tr}\hat{P}_{\mathrm{e}}^2 = (1/4)(w/\hbar\pi^{1/2})^2(w/\hbar\pi^{1/2})^2 \int \mathrm{d}^2p'' \exp(-p''^2w^2/2\hbar^2) \times \int \mathrm{d}^2p'[L_{n_0}(|\sigma(\vec{p}')|^2)]^2 \exp(-|\sigma(\vec{p}')|^2) \exp(-p'^2w^2/2\hbar^2), \tag{B.26}$$

where we have used the explicit form $g(\vec{p}) = (w/\hbar\pi^{1/2}) \exp(-p^2w^2/2\hbar^2)$ of the momentum profile function introduced in the main text in equation (18). After having performed the $\vec{p}''$-integration, and introducing dimensionless parameters and integration variables, we have from equation (26)

$$\begin{aligned} \mathrm{Tr}\vec{P}_{\mathrm{e}}^2 &= \int \mathrm{d}x\, x[L_{n_0}(b^2x^2)]^2 \exp[-(b^2+1/2)x^2] \\ &= \frac{1}{2b^2} \cdot \left(1 - \frac{1}{2b^2}\right)^{n_0} P_{n_0}\left[\frac{1+(1/2b^2)^2}{1-(1/2b^2)}\right], \end{aligned} \tag{B.27}$$

where $P_n(x)$ are the Legendre polynomials of the order n. In obtaining equation (B.27), we have used the definitions in equation (A.3) and the formula 7.414.2 of Gradshteyn and Ryzhik (2000). According to equation (B.27) and the general formula given by equation (20), the exact expression for the electron's linear entropy can be written as

$$\begin{aligned} H_{\mathrm{electron}}[\hat{P}_{\mathrm{e}}] &= 1 - \frac{1}{2b^2} \cdot \left(1 - \frac{1}{2b^2}\right)^{n_0} P_{n_0}\left[\frac{1+(1/2b^2)^2}{1-(1/2b^2)}\right], \\ b &\equiv \frac{ea}{mc\hbar\Omega\sqrt{2}}\frac{\hbar}{w} = \frac{1}{\sqrt{n_0}}\frac{1}{2}(\mu\Lambda/w), \end{aligned} \tag{B.28}$$

where the definitions in equation (A.9) have also been used. In the large n_0 limit we can use Hilb's formula, equation (A.14), and apply it directly in the integrand in equation (B.27). We have

$$\begin{aligned} &\mathrm{Tr}\hat{P}_{\mathrm{e}}^2 = \int_0^\infty \mathrm{d}x\, xJ_0^2[(\mu\Lambda/w)x] \exp(-x^2/2) + O(n_0^{-3/4}) = I_0(2q)\mathrm{e}^{-2q} + O(n_0^{-3/4}), \\ &q \equiv \tfrac{1}{2}(\mu\Lambda/w)^2. \end{aligned} \tag{B.29}$$

In obtaining the above analytic result we have used Weber's second exponential integral formula already displayed in equation (A.17). By using equation (B.29), the electron's linear entropy in the large n_0 limit turns out to be

$$H_{\mathrm{electron}}[\hat{P}_{\mathrm{e}}] = 1 - I_0(2q)\mathrm{e}^{-2q}, \tag{B.30}$$

which coincides with the linear entropy of the photon field given by equation (B.21),

$$H_{\mathrm{electron}}[\hat{P}_{\mathrm{e}}] = H_{\mathrm{photon}}[\{p_k\}] = 1 - I_0(2q)\mathrm{e}^{-2q}. \tag{B.31}$$

At the end of the present appendix, we give an estimate for the average occupation number of the photon field expressed in terms of the amplitude of the electric field strength F_0 or, equivalently, in terms of the intensity I of a quasi-monochromatic radiation. In free space, the three-dimensional spatial-mode density in a frequency interval $(\omega, \omega + \Delta\omega)$ can be expressed as $Z_\omega\Delta\omega$, where $Z_\omega = \omega^2/\pi^2c^3$ is the spectral mode density. We equate the time average of

Table B.1. For three kinds of lasers, the numerical values of the photon energy, the inverse bandwidth and the corresponding average photon occupation number expressed in terms of the dimensionless intensity I.

	5.2 fs Ti : Sa laser	5.2 ps Nd : glass laser	CW He–Ne laser
E_{ph}	1.57	1.17	1.96
$\omega/\Delta\omega$	2	3×10^3	10^8
$\bar{n}$	$9 \times 10^{-6}\, I$	$6 \times 10^{-2}\, I$	$2 \times 10^3\, I$

the energy density $F_0^2/4\pi$ of the radiation with the mode density $Z_\omega \Delta\omega$ times the average occupation number $\bar{n}$ of the modes multiplied by the central photon energy $\hbar\omega$, i.e.

$$Z_\omega \Delta\omega \cdot \bar{n} \cdot \hbar\omega = F_0^2/4\pi \rightarrow \bar{n} = \frac{\pi^2 c^3}{\omega^2} \cdot \frac{1}{\Delta\omega\hbar\omega} \cdot \frac{F_0^2}{4\pi}. \tag{B.32}$$

By introducing dimensionless combinations of the parameters, we obtain from equation (B.32)

$$\bar{n} = \frac{\pi}{16}\left(\frac{\hbar c}{e^2}\right) \cdot \left(\frac{2mc^2}{\hbar\omega}\right)^2 \cdot \left(\frac{\omega}{\Delta\omega}\right) \cdot \left(\frac{eF_0}{mc\omega}\right)^2 = \frac{\pi}{16\alpha} \cdot \left(\frac{2mc^2}{\hbar\omega}\right)^2 \cdot \left(\frac{\omega}{\Delta\omega}\right) \cdot \mu^2, \tag{B.33}$$

where $\alpha \equiv e^2/\hbar c \approx 1/137$ denotes the fine structure constant and $\mu \equiv eF_0/mc\omega$ is the dimensionless intensity parameter already introduced in equation (23) of section 4. The second factor on the right-hand side of the last equation in equation (B.33) is the square of the ratio of the electron–positron pair creation energy $2mc^2 \approx 10^6$ eV to the photon energy $\hbar\omega$. For optical fields, this factor is of the order of 10^{12}, because in this case, $\hbar\omega \approx 1$ eV. The third factor $\omega/\Delta\omega$ is the inverse relative bandwidth of the radiation, which, in the case of pulsed lasers, is of the order of τ_{pulse}/T, where τ_{pulse} is the pulse duration and T is the period of the central spectral component. On the basis of equation (B.33), the numerical value of $\bar{n}$ can be calculated according to the formula

$$\bar{n} = 2.7 \times 10^{-5}(\omega/\Delta\omega) \cdot (I/E_{\mathrm{ph}}^4), \tag{B.34}$$

where I denotes the intensity divided by W cm^{-2}, and E_{ph} is the photon energy measured in eV s. In table B.1, we summarize for three kinds of lasers, the numerical values of the photon energy, the inverse bandwidth and the corresponding average photon occupation number, expressed in terms of the dimensionless intensity I, on the basis of equation (B.34).

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